

# Introduction to Laplace's Equation

MATH 467 *Partial Differential Equations*

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Fall 2022

# Objectives

In this lesson we will learn about

- ▶ the partial differential equations and boundary value problems known as Laplace's and Poisson's equations,
- ▶ techniques for solving Laplace's and Poisson's equations, and
- ▶ applications of Laplace's and Poisson's equations.

# The Laplacian Operator

Let  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  and let function  $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , then the **Laplacian** of  $u$  is

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}.$$

If  $n = 2$  we will often write  $\mathbf{x} = (x, y)$  and

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

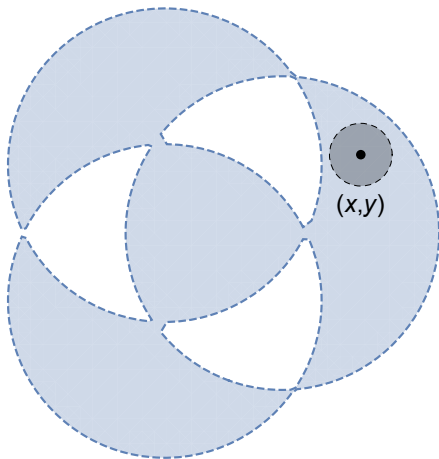
If  $n = 3$  we will often write  $\mathbf{x} = (x, y, z)$  and

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

# Open Sets

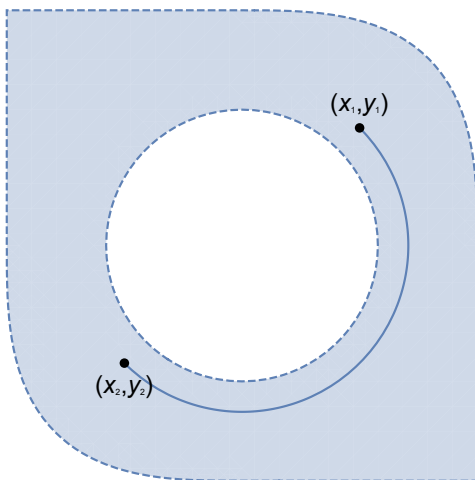
A set  $\Omega \subset \mathbb{R}^n$  is **open** if for every  $\mathbf{x} \in \Omega$  there exists  $\delta > 0$  such that

$$\{\mathbf{y} \mid \|\mathbf{x} - \mathbf{y}\| < \delta\} \subset \Omega.$$



# Connected Sets

A set  $\Omega \subset \mathbb{R}^n$  is **connected** if for every  $\mathbf{x}, \mathbf{y} \in \Omega$  there exists a path connecting  $\mathbf{x}$  and  $\mathbf{y}$  which lies completely within  $\Omega$ .



# Laplace's Equation

If  $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$  where  $\Omega$  is an open, connected set then  $u$  satisfies **Laplace's equation** if

$$\Delta u = 0.$$

Function  $u$  is said to be a **harmonic function**.

If function  $f$  is defined on  $\Omega$  then  $u$  satisfies **Poisson's equation** if

$$\Delta u = f.$$

## Example

Let  $u(x, y) = e^x \cos y$  and show  $u$  is harmonic on  $\Omega = \mathbb{R}^2$ .

$$\Delta u = e^x \cos y + (e^x(-\cos y)) = 0$$

for all  $(x, y) \in \Omega$ .

# Boundary Conditions

- ▶ If  $\Omega \subset \mathbb{R}^n$  is an open set then the set of boundary points of  $\Omega$  will be denoted  $\partial\Omega$ .
- ▶ Laplace's and Poisson's equations must be supplemented with boundary conditions.

Dirichlet BCs:  $u(\mathbf{x}) = \phi(\mathbf{x})$  for  $\mathbf{x} \in \partial\Omega$ .

Neumann BCs:  $\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = \phi(\mathbf{x})$  for  $\mathbf{x} \in \partial\Omega$  where  $\mathbf{n}$  is the unit outward normal vector to  $\partial\Omega$ .

Mixed BCs:  $u(\mathbf{x}) + \alpha \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = \phi(\mathbf{x})$  for  $\mathbf{x} \in \partial\Omega$  where  $\alpha$  is constant and  $\mathbf{n}$  is the unit outward normal vector to  $\partial\Omega$ .

# Dirichlet Problem on a Rectangle

Define

$$\Omega = \{(x, y) \mid 0 < x < a, 0 < y < b\} = (0, a) \times (0, b)$$

where  $a$  and  $b$  are constants. In this case

$$\partial\Omega = \{(x, y) \mid x = 0, a \text{ and } 0 \leq y \leq b, \text{ or } y = 0, b \text{ and } 0 \leq x \leq a\}.$$

Consider the following example of Laplace's equation.

$$\Delta u = 0 \quad \text{for } (x, y) \in \Omega$$

$$u(0, y) = 0$$

$$u(a, y) = f(y)$$

$$u(x, 0) = 0$$

$$u(x, b) = 0$$

This problem can be solved using separation of variables and Fourier series.

# Separation of Variables

Assume  $u(x, y) = X(x)Y(y)$ , then

$$\begin{aligned}X''(x)Y(y) + X(x)Y''(y) &= 0 \\ \frac{X''(x)Y(y)}{X(x)Y(y)} + \frac{X(x)Y''(y)}{X(x)Y(y)} &= 0 \\ -\frac{X''(x)}{X(x)} &= \frac{Y''(y)}{Y(y)} = c\end{aligned}$$

where  $c$  is a constant.

The boundary conditions become

$$\begin{aligned}u(0, y) = X(0)Y(y) = 0 &\implies X(0) = 0 \\ u(x, 0) = X(x)Y(0) = 0 &\implies Y(0) = 0 \\ u(x, b) = X(x)Y(b) = 0 &\implies Y(b) = 0.\end{aligned}$$

## Boundary Value Problem for $Y(y)$

Separation of variables and the Dirichlet boundary conditions imply the following BVP for  $Y(y)$ :

$$\begin{aligned}Y(y) - c Y(y) &= 0 \quad \text{for } 0 < y < b \\Y(0) &= 0 \\Y(b) &= 0.\end{aligned}$$

Previously we have seen the only nontrivial solutions occur when

$$c = -\lambda_n^2 = -\frac{n^2\pi^2}{b^2}$$

and  $Y_n(y) = \sin \frac{n\pi y}{b}$  for  $n \in \mathbb{N}$ .

## Boundary Value Problem for $X(x)$

Function  $X(x)$  must satisfy the following boundary value problem.

$$\begin{aligned}X''(x) + c X(x) &= X''(x) - \frac{n^2 \pi^2}{b^2} X(x) = 0 \\ X(0) &= 0\end{aligned}$$

The general solution takes the form

$$X(x) = A \cosh \frac{n\pi x}{b} + B \sinh \frac{n\pi x}{b}$$

and  $A = 0$  in order to satisfy the boundary condition. Thus

$$X_n(x) = \sinh \frac{n\pi x}{b}.$$

# Product Solution

- ▶ Define the function

$$u_n(x, y) = \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

for  $n \in \mathbb{N}$ .

- ▶ Each  $u_n(x, y)$  satisfies Laplace's equation on the rectangle and the boundary conditions for  $(x, y) = (x, 0)$  and  $(x, y) = (x, b)$  for  $0 \leq x \leq a$  and for  $(x, y) = (0, y)$  for  $0 \leq y \leq b$ .
- ▶ By the Principle of Superposition

$$u(x, y) = \sum_{n=1}^N c_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

will also satisfy these equations for any  $N \geq 1$  and constants  $c_n$ .

## Boundary $(a, y)$ for $0 \leq y \leq b$

- ▶ Extend  $f(y)$  as an odd,  $2b$ -periodic function.
- ▶ For  $0 \leq y \leq b$

$$\begin{aligned} f(y) &= u(a, y) \\ &= \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b}. \end{aligned}$$

- ▶ Multiply both sides of the equation by  $\sin \frac{m\pi y}{b}$  and integrate over  $[0, b]$ .

$$\begin{aligned} \int_0^b f(y) \sin \frac{m\pi y}{b} dy &= \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi a}{b} \int_0^b \sin \frac{n\pi y}{b} \sin \frac{m\pi y}{b} dy \\ &= \frac{b}{2} c_m \sinh \frac{m\pi a}{b} \\ c_m &= \frac{2}{b \sinh \frac{m\pi a}{b}} \int_0^b f(y) \sin \frac{m\pi y}{b} dy \end{aligned}$$

# General Solution

The solution to

$$\Delta u = 0 \quad \text{for } (x, y) \in \Omega$$

$$u(0, y) = 0$$

$$u(a, y) = f(y)$$

$$u(x, 0) = 0$$

$$u(x, b) = 0$$

is therefore

$$u(x, y) = \sum_{n=1}^{\infty} \left( \frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b f(y) \sin \frac{n\pi y}{b} dy \right) \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}.$$

# General Dirichlet Problem on a Rectangle

The general boundary value problem:

$$\Delta u = 0 \quad \text{for } (x, y) \in \Omega$$

$$u(0, y) = g_1(y)$$

$$u(a, y) = g_2(y)$$

$$u(x, 0) = f_1(x)$$

$$u(x, b) = f_2(x)$$

can be decomposed into four simpler boundary value problems.

# Decomposition

For  $(x, y) \in \Omega$ ,

$$\Delta u_1 = 0$$

$$\Delta u_2 = 0$$

$$\Delta u_3 = 0$$

$$\Delta u_4 = 0$$

$$u_1(0, y) = g_1(y)$$

$$u_2(0, y) = 0$$

$$u_3(0, y) = 0$$

$$u_4(0, y) = 0$$

$$u_1(a, y) = 0$$

$$u_2(a, y) = g_2(y)$$

$$u_3(a, y) = 0$$

$$u_4(a, y) = 0$$

$$u_1(x, 0) = 0$$

$$u_2(x, 0) = 0$$

$$u_3(x, 0) = f_1(x)$$

$$u_4(x, 0) = 0$$

$$u_1(x, b) = 0$$

$$u_2(x, b) = 0$$

$$u_3(x, b) = 0$$

$$u_4(x, b) = f_2(x)$$

If  $u_i(x, y)$  solves the  $i$ th boundary value problem, then

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

solves the general Dirichlet boundary value problem on  $\Omega$ .

# Solutions

$$u_1(x, y) = \sum_{n=1}^{\infty} \left( \frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b g_1(y) \sin \frac{n\pi y}{b} dy \right) \sinh \frac{n\pi(a-x)}{b} \sin \frac{n\pi y}{b}$$

$$u_2(x, y) = \sum_{n=1}^{\infty} \left( \frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b g_2(y) \sin \frac{n\pi y}{b} dy \right) \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

$$u_3(x, y) = \sum_{n=1}^{\infty} \left( \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f_1(x) \sin \frac{n\pi x}{a} dx \right) \sinh \frac{n\pi(b-y)}{a} \sin \frac{n\pi x}{a}$$

$$u_4(x, y) = \sum_{n=1}^{\infty} \left( \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f_2(x) \sin \frac{n\pi x}{a} dx \right) \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a}$$

## Example

Let  $\Omega = (0, 1) \times (0, 1)$  and solve the Dirichlet boundary value problem for Laplace's equation on  $\Omega$ .

$$\Delta u = 0 \quad \text{for } (x, y) \in \Omega$$

$$u(0, y) = 0$$

$$u(1, y) = 1 - y$$

$$u(x, 0) = x$$

$$u(x, 1) = 0$$

## Solution

Let  $a = b = 1$ , then using the notation prescribed earlier the solution can be expressed as

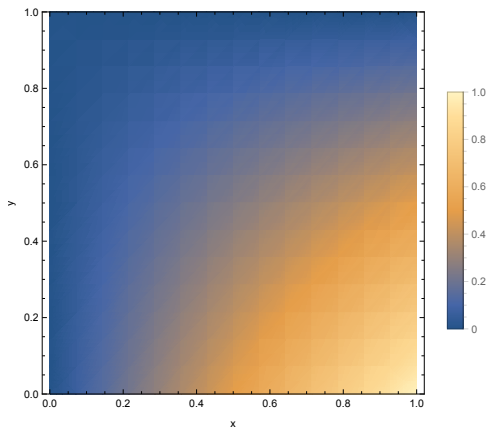
$$u(x, y) = u_2(x, y) + u_3(x, y)$$

where

$$\begin{aligned} u_2(x, y) &= \sum_{n=1}^{\infty} \left( \frac{2}{\sinh(n\pi)} \int_0^1 (1-y) \sin(n\pi y) dy \right) \sinh(n\pi x) \sin(n\pi y) \\ &= \sum_{n=1}^{\infty} \frac{2}{n\pi \sinh(n\pi)} \sinh(n\pi x) \sin(n\pi y) \\ u_3(x, y) &= \sum_{n=1}^{\infty} \left( \frac{2}{\sinh(n\pi)} \int_0^1 x \sin(n\pi x) dx \right) \sinh(n\pi(1-y)) \sin(n\pi x) \\ &= - \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi \sinh(n\pi)} \sinh(n\pi(1-y)) \sin(n\pi x) \end{aligned}$$

# Illustration

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2}{n\pi \sinh(n\pi)} \sinh(n\pi x) \sin(n\pi y) - \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi \sinh(n\pi)} \sinh(n\pi(1-y)) \sin(n\pi x)$$



# Homework

- ▶ Read Section 6.1–6.2
- ▶ Exercises: 1–7