Introduction to Laplace's Equation

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will learn about

- the partial differential equations and boundary value problems known as Laplace's and Poisson's equations,
- techniques for solving Laplace's and Poisson's equations, and
- applications of Laplace's and Poisson's equations.

The Laplacian Operator

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and let function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, then the **Laplacian** of u is

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}.$$

If $n = 2$ we will often write $\mathbf{x} = (x, y)$ and

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

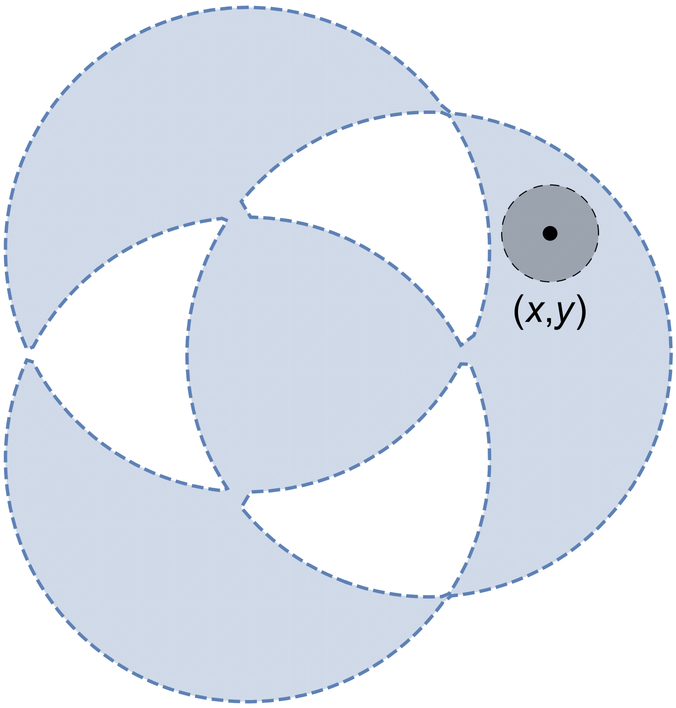
If $n = 3$ we will often write $\mathbf{x} = (x, y, z)$ and

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Open Sets

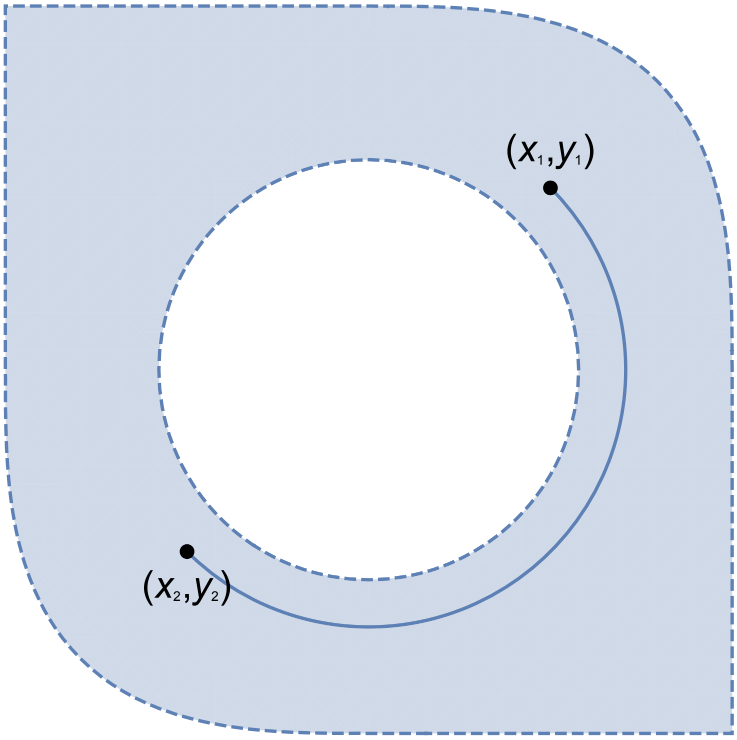
A set $\Omega \subset \mathbb{R}^n$ is **open** if for every $\mathbf{x} \in \Omega$ there exists $\delta > 0$ such that

$$\{\mathbf{y} \mid \|\mathbf{x} - \mathbf{y}\| < \delta\} \subset \Omega.$$



Connected Sets

A set $\Omega \subset \mathbb{R}^n$ is **connected** if for every $\mathbf{x}, \mathbf{y} \in \Omega$ there exists a path connecting $\mathbf{x}$ and $\mathbf{y}$ which lies completely within Ω.



Laplace's Equation

If $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ where Ω is an open, connected set then u satisfies **Laplace's equation** if

$$\Delta u = 0.$$

Function u is said to be a **harmonic function**.

If function f is defined on Ω then u satisfies **Poisson's equation** if

$$\Delta u = f.$$

Example

Let $u(x,y) = e^x \cos y$ and show u is harmonic on $\Omega = \mathbb{R}^2$.

$$\Delta u = e^x \cos y + (e^x(-\cos y)) = 0$$

for all $(x,y) \in \Omega$.

Boundary Conditions

- If $\Omega \subset \mathbb{R}^n$ is an open set then the set of boundary points of Ω will be denoted $\partial\Omega$.
- Laplace's and Poisson's equations must be supplemented with boundary conditions.

Dirichlet BCs: $u(\mathbf{x}) = \phi(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$.

Neumann BCs: $\dfrac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = \phi(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$ where $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$.

Mixed BCs: $u(\mathbf{x}) + \alpha \dfrac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = \phi(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$ where α is constant and $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$.

Dirichlet Problem on a Rectangle

Define

$$\Omega = \{(x, y) \,|\, 0 < x < a,\ 0 < y < b\} = (0, a) \times (0, b)$$

where a and b are constants. In this case

$$\partial\Omega = \{(x, y) \,|\, x = 0,\ a \text{ and } 0 \le y \le b, \text{ or } y = 0,\ b \text{ and } 0 \le x \le a\}.$$

Consider the following example of Laplace's equation.

$$\begin{aligned} \Delta u &= 0 \quad \text{for } (x, y) \in \Omega \\ u(0, y) &= 0 \\ u(a, y) &= f(y) \\ u(x, 0) &= 0 \\ u(x, b) &= 0 \end{aligned}$$

This problem can be solved using separation of variables and Fourier series.

Separation of Variables

Assume $u(x, y) = X(x)Y(y)$, then

$$\begin{aligned} X''(x)Y(y) + X(x)Y''(y) &= 0 \\ \frac{X''(x)Y(y)}{X(x)Y(y)} + \frac{X(x)Y''(y)}{X(x)Y(y)} &= 0 \\ -\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} &= c \end{aligned}$$

where c is a constant.

The boundary conditions become

$$\begin{aligned} u(0, y) &= X(0)Y(y) = 0 \implies X(0) = 0 \\ u(x, 0) &= X(x)Y(0) = 0 \implies Y(0) = 0 \\ u(x, b) &= X(x)Y(b) = 0 \implies Y(b) = 0. \end{aligned}$$

Boundary Value Problem for $Y(y)$

Separation of variables and the Dirichlet boundary conditions imply the following BVP for $Y(y)$:

$$
\begin{aligned}
Y(y) - c\,Y(y) &= 0 \quad \text{for } 0 < y < b \\
Y(0) &= 0 \\
Y(b) &= 0.
\end{aligned}
$$

Previously we have seen the only nontrivial solutions occur when

$$c = -\lambda_n^2 = -\frac{n^2\pi^2}{b^2}$$

and $Y_n(y) = \sin \dfrac{n\pi y}{b}$ for $n \in \mathbb{N}$.

Boundary Value Problem for $X(x)$

Function $X(x)$ must satisfy the following boundary value problem.

$$
\begin{aligned}
X''(x) + c\,X(x) = X''(x) - \frac{n^2\pi^2}{b^2} X(x) &= 0 \\
X(0) &= 0
\end{aligned}
$$

The general solution takes the form

$$X(x) = A \cosh \frac{n\pi x}{b} + B \sinh \frac{n\pi x}{b}$$

and $A = 0$ in order to satisfy the boundary condition. Thus

$$X_n(x) = \sinh \frac{n\pi x}{b}.$$

Product Solution

- Define the function

$$u_n(x, y) = \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

for $n \in \mathbb{N}$.
- Each $u_n(x, y)$ satisfies Laplace's equation on the rectangle and the boundary conditions for $(x, y) = (x, 0)$ and $(x, y) = (x, b)$ for $0 \leq x \leq a$ and for $(x, y) = (0, y)$ for $0 \leq y \leq b$.
- By the Principle of Superposition

$$u(x, y) = \sum_{n=1}^{N} c_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

will also satisfy these equations for any $N \geq 1$ and constants c_n.

Boundary (a, y) for $0 \leq y \leq b$

- Extend $f(y)$ as an odd, $2b$-periodic function.
- For $0 \leq y \leq b$

$$\begin{aligned} f(y) &= u(a, y) \\ &= \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b}. \end{aligned}$$

- Multiply both sides of the equation by $\sin \frac{m\pi y}{b}$ and integrate over $[0, b]$.

$$\begin{aligned} \int_0^b f(y) \sin \frac{m\pi y}{b}\, dy &= \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi a}{b} \int_0^b \sin \frac{n\pi y}{b} \sin \frac{m\pi y}{b}\, dy \\ &= \frac{b}{2} c_m \sinh \frac{m\pi a}{b} \\ c_m &= \frac{2}{b \sinh \frac{m\pi a}{b}} \int_0^b f(y) \sin \frac{m\pi y}{b}\, dy \end{aligned}$$

General Solution

The solution to

$$\begin{aligned} \Delta u &= 0 \quad \text{for } (x,y) \in \Omega \\ u(0,y) &= 0 \\ u(a,y) &= f(y) \\ u(x,0) &= 0 \\ u(x,b) &= 0 \end{aligned}$$

is therefore

$$u(x,y) = \sum_{n=1}^{\infty} \left(\frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy \right) \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}.$$

General Dirichlet Problem on a Rectangle

The general boundary value problem:

$$\begin{aligned} \Delta u &= 0 \quad \text{for } (x,y) \in \Omega \\ u(0,y) &= g_1(y) \\ u(a,y) &= g_2(y) \\ u(x,0) &= f_1(x) \\ u(x,b) &= f_2(x) \end{aligned}$$

can be decomposed into four simpler boundary value problems.

Decomposition

For $(x, y) \in \Omega$,

$$
\begin{array}{cccc}
\Delta u_1 = 0 & \Delta u_2 = 0 & \Delta u_3 = 0 & \Delta u_4 = 0 \\
u_1(0, y) = g_1(y) & u_2(0, y) = 0 & u_3(0, y) = 0 & u_4(0, y) = 0 \\
u_1(a, y) = 0 & u_2(a, y) = g_2(y) & u_3(a, y) = 0 & u_4(a, y) = 0 \\
u_1(x, 0) = 0 & u_2(x, 0) = 0 & u_3(x, 0) = f_1(x) & u_4(x, 0) = 0 \\
u_1(x, b) = 0 & u_2(x, b) = 0 & u_3(x, b) = 0 & u_4(x, b) = f_2(x)
\end{array}
$$

If $u_i(x, y)$ solves the ith boundary value problem, then

$$u(x, y) = u_1(x, t) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

solves the general Dirichlet boundary value problem on Ω.

Solutions

$$u_1(x, y) = \sum_{n=1}^{\infty} \left(\frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b g_1(y) \sin \frac{n\pi y}{b}\, dy \right) \sinh \frac{n\pi(a-x)}{b} \sin \frac{n\pi y}{b}$$

$$u_2(x, y) = \sum_{n=1}^{\infty} \left(\frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b g_2(y) \sin \frac{n\pi y}{b}\, dy \right) \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

$$u_3(x, y) = \sum_{n=1}^{\infty} \left(\frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f_1(x) \sin \frac{n\pi x}{a}\, dx \right) \sinh \frac{n\pi(b-y)}{a} \sin \frac{n\pi x}{a}$$

$$u_4(x, y) = \sum_{n=1}^{\infty} \left(\frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f_2(x) \sin \frac{n\pi x}{a}\, dx \right) \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a}$$

Example

Let $\Omega = (0,1) \times (0,1)$ and solve the Dirichlet boundary value problem for Laplace's equation on Ω.

$$\begin{aligned} \Delta u &= 0 \quad \text{for } (x,y) \in \Omega \\ u(0,y) &= 0 \\ u(1,y) &= 1 - y \\ u(x,0) &= x \\ u(x,1) &= 0 \end{aligned}$$

Solution

Let $a = b = 1$, then using the notation prescribed earlier the solution can be expressed as

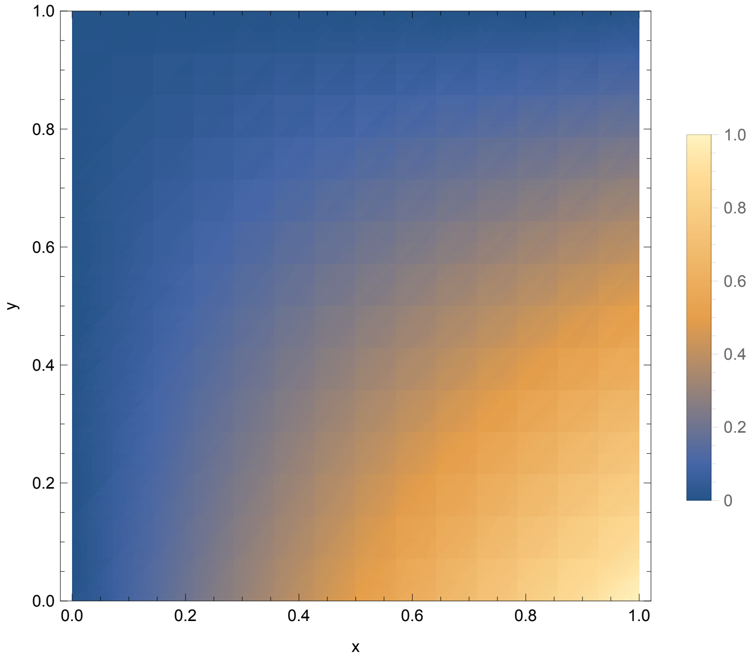
$$u(x,y) = u_2(x,y) + u_3(x,y)$$

where

$$\begin{aligned} u_2(x,y) &= \sum_{n=1}^{\infty} \left(\frac{2}{\sinh(n\pi)} \int_0^1 (1-y)\sin(n\pi y)\,dy \right) \sinh(n\pi x)\sin(n\pi y) \\ &= \sum_{n=1}^{\infty} \frac{2}{n\pi\sinh(n\pi)} \sinh(n\pi x)\sin(n\pi y) \\ u_3(x,y) &= \sum_{n=1}^{\infty} \left(\frac{2}{\sinh(n\pi)} \int_0^1 x\sin(n\pi x)\,dx \right) \sinh(n\pi(1-y))\sin(n\pi x) \\ &= -\sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi\sinh(n\pi)} \sinh(n\pi(1-y))\sin(n\pi x) \end{aligned}$$

Illustration

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{n\pi \sinh(n\pi)} \sinh(n\pi x)\sin(n\pi y) - \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi \sinh(n\pi)} \sinh(n\pi(1-y))\sin(n\pi x)$$



Homework

- Read Section 6.1–6.2
- Exercises: 1–7