

# Laplace's Equation on a Disk

MATH 467 *Partial Differential Equations*

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## Laplace's Equation on a Disk

Consider the Dirichlet boundary value problem:

$$\begin{aligned}\Delta u &= 0 \text{ for } x^2 + y^2 < a^2 \\ u(x, y) &= \phi(x, y) \text{ for } x^2 + y^2 = a^2.\end{aligned}$$

**Remark:** since the boundary of  $\Omega$  is not a rectangle, we cannot use separation of variables directly. Instead we must convert to polar coordinates before using separation of variables.

In polar coordinates  $(r, \theta)$  the Laplacian operator can be expressed as

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta},$$

for  $r > 0$ .

# Dirichlet BVP in Polar Coordinates

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \text{ for } 0 < r < a \text{ and } -\infty < \theta < \infty$$
$$u(a, \theta) = f(\theta) = \phi(a \cos \theta, a \sin \theta)$$

## Remarks:

- ▶ The boundary conditions are periodic in  $\theta$  with period  $2\pi$ .
- ▶ The solution  $u(r, \theta)$  should be  $2\pi$ -periodic in  $\theta$ .
- ▶ The Laplacian is not defined for  $r = 0$ , but we wish for the solution to remain finite as  $r \rightarrow 0^+$ .

# Separation of Variables in Polar Coordinates

Assume  $u(r, \theta) = R(r)T(\theta)$ , then

$$R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) + \frac{1}{r^2}R(r)T''(\theta) = 0$$
$$\frac{R''(r)T(\theta)}{R(r)T(\theta)} + \frac{1}{r}\frac{R'(r)T(\theta)}{R(r)T(\theta)} + \frac{1}{r^2}\frac{R(r)T''(\theta)}{R(r)T(\theta)} = 0$$
$$\frac{r^2 R''(r)}{R(r)} + \frac{r R'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} = c$$

where  $c$  is a constant.

# Implied Ordinary Differential Equations

$$T''(\theta) + c T(\theta) = 0$$

case  $c = 0$ : the only nontrivial  $2\pi$ -periodic solution is

$$T_0(\theta) = A_0$$

case  $c = \lambda^2 > 0$ : the only nontrivial  $2\pi$ -periodic solution is

$$T_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$$

and  $c = \lambda_n^2 = n^2$  for  $n \in \mathbb{N}$ .

## Euler's Equation

The implied ordinary differential equation for  $R(r)$  is

$$r^2 R''(r) + r R'(r) - c R(r) = 0,$$

which is known as **Euler's equation**.

case  $c = 0$ :

$$0 = r^2 R''(r) + r R'(r)$$

$$R_0(r) = C_0 \ln r + D_0$$

case  $c = n^2 > 0$ :

$$0 = r^2 R''(r) + r R'(r) - n^2 R(r)$$

$$R_n(r) = C_n r^{-n} + D_n r^n$$

In order for the product solution to be bounded as  $r \rightarrow 0^+$  we must choose  $C_0 = C_1 = \dots = 0$ .

# Product Solution

Define the function

$$u_n(r, \theta) = R_n(r) T_n(\theta) = r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

for  $n = 0, 1, \dots$

By the Principle of Superposition the function

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

satisfies the Laplacian on the disk for any choice of constants  $A_0$ ,  $A_n$ , and  $B_n$ .

Fourier series techniques can be used to satisfy the boundary condition.

## Boundary Condition

Since  $u(a, \theta) = f(\theta)$  then

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} a^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta = \frac{\alpha_0}{2}$$

$$a^n A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta = \alpha_n$$

$$a^n B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta = \beta_n$$

and

$$u(r, \theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n [\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)].$$

## Example

Solve Laplace's equation on the unit disk with the following Dirichlet boundary condition.

$$\begin{aligned}\Delta u &= 0 \text{ for } x^2 + y^2 < 1 \\ u(1, \theta) &= \pi - \theta \text{ for } -\pi < \theta < \pi\end{aligned}$$

## Solution

The solution takes the form

$$u(r, \theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n [\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)]$$

where

$$\alpha_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - \theta) d\theta = 2\pi$$

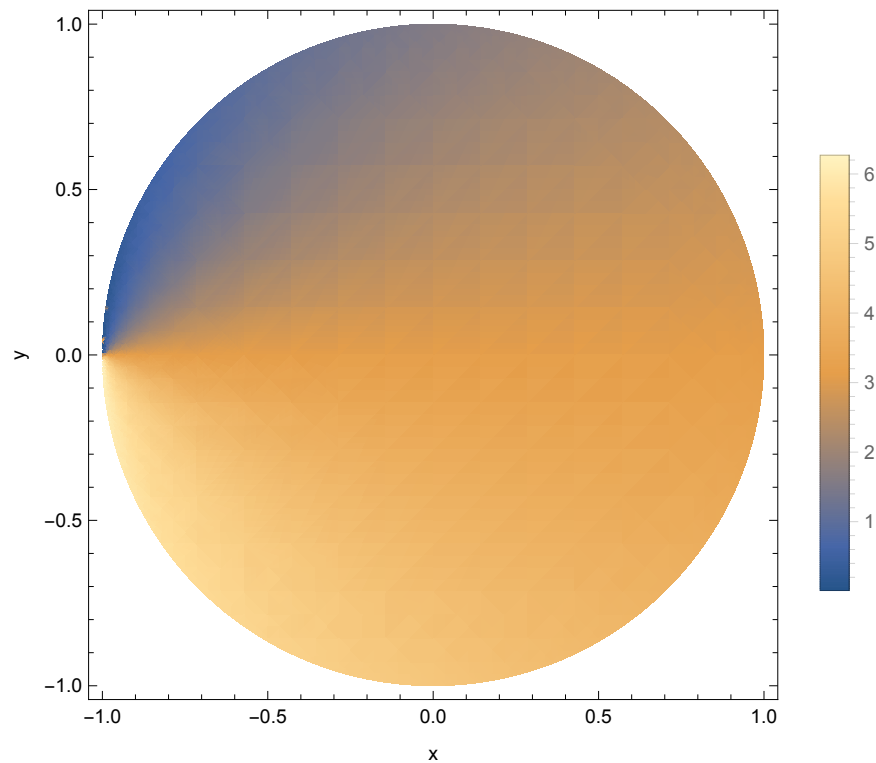
$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - \theta) \cos(n\theta) d\theta = 0$$

$$\beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - \theta) \sin(n\theta) d\theta = \frac{2(-1)^n}{n}.$$

$$u(r, \theta) = \pi + \sum_{n=1}^{\infty} \frac{2(-1)^n r^n}{n} \sin(n\theta)$$

# Illustration

$$u(r, \theta) = \pi + \sum_{n=1}^{\infty} \frac{2(-1)^n r^n}{n} \sin(n\theta)$$



## Complex Arithmetic (1 of 3)

Suppose  $z = a + ib$  where  $a, b \in \mathbb{R}$  and  $i = \sqrt{-1}$ .

$$\sum_{n=0}^{\infty} (-z)^n = \frac{1}{1+z} \quad (\text{if } |z| < 1)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1} = \ln(1+z)$$

$$-\sum_{n=1}^{\infty} \frac{(-z)^n}{n} = \ln(1+z)$$

## Complex Arithmetic (2 of 3)

In polar coordinate form  $z = r e^{i\theta}$  where  $r = \sqrt{a^2 + b^2} = |z|$  and  $\theta = \tan^{-1}(b/a)$ .

$$\begin{aligned} -\sum_{n=1}^{\infty} \frac{(-z)^n}{n} &= \ln(1+z) \\ -\sum_{n=1}^{\infty} \frac{1}{n} (-re^{i\theta})^n &= \ln(1+re^{i\theta}) \text{ (if } |r| < 1) \\ -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n e^{in\theta} &= \ln(1+re^{i\theta}) \\ -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \operatorname{Im}(e^{in\theta}) &= \operatorname{Im}(\ln(1+re^{i\theta})) \\ -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta) &= \operatorname{Im}(\ln(1+r\cos\theta + ir\sin\theta)) \end{aligned}$$

## Natural Logarithm of a Complex Number

Suppose  $z = a + ib \in \mathbb{C}$  and  $w = \alpha + i\beta \in \mathbb{C}$  such that

$$e^w = z$$

then  $w = \ln z$ .

$$\begin{aligned} e^{\alpha+i\beta} &= a + ib \\ e^{\alpha} e^{i\beta} &= a + ib \\ e^{\alpha}(\cos\beta + i\sin\beta) &= a + ib \end{aligned}$$

Thus  $a = e^{\alpha} \cos\beta$  and  $b = e^{\alpha} \sin\beta$  and

$$\begin{aligned} a^2 + b^2 &= e^{2\alpha} \implies \alpha = \ln \sqrt{a^2 + b^2} \\ \frac{b}{a} &= \tan\beta \implies \beta = \tan^{-1} \frac{b}{a}. \end{aligned}$$

Thus

$$\ln z = \ln \sqrt{a^2 + b^2} + i \tan^{-1} \frac{b}{a}.$$

## Complex Arithmetic (3 of 3)

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta) = \operatorname{Im} (\ln(1 + r \cos \theta + ir \sin \theta))$$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta) = \tan^{-1} \left( \frac{r \sin \theta}{1 + r \cos \theta} \right)$$

$$\begin{aligned} u(r, \theta) &= \pi + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta) \\ &= \pi - 2 \tan^{-1} \left( \frac{r \sin \theta}{1 + r \cos \theta} \right) \end{aligned}$$

## Laplace's Equation on a Sector of an Annulus

Find the solution to the following boundary value problem.

$$\Delta u = 0 \text{ for } 1 < x^2 + y^2 < 4 \text{ with } x > 0 \text{ and } y > 0$$

$$u(x, 0) = 0$$

$$u(0, y) = 0$$

$$u(x, y) = 2xy \text{ for } x^2 + y^2 = 1$$

$$u(x, y) = \left( \frac{\pi}{2} - \tan^{-1} \frac{y}{x} \right) \tan^{-1} \frac{y}{x} \text{ for } x^2 + y^2 = 4$$



# Polar Coordinates

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \text{ for } 1 < r < 2 \text{ with } 0 < \theta < \pi/2$$

$$u(r, 0) = 0$$

$$u(r, \pi/2) = 0$$

$$u(1, \theta) = \sin(2\theta)$$

$$u(2, \theta) = \theta \left( \frac{\pi}{2} - \theta \right)$$

We will again use separation of variables.

## Separation of Variables

Assuming  $u(r, \theta) = R(r)T(\theta)$  then

$$R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) + \frac{1}{r^2}R(r)T''(\theta) = 0$$

$$\frac{r^2 R''(r)}{R(r)} + \frac{r R'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} = c.$$

This implies the following boundary value problem for the angular factor of the solution.

$$T''(\theta) + cT(\theta) = 0$$

$$T(0) = 0$$

$$T(\pi/2) = 0$$

# Eigenfunctions and Eigenvalues

The only nontrivial solutions to

$$T''(\theta) + cT(\theta) = 0$$

$$T(0) = 0$$

$$T(\pi/2) = 0$$

are

$$T_n(\theta) = \sin(2n\theta)$$

with  $c = \lambda_n^2 = 4n^2$  for  $n \in \mathbb{N}$ .

Using the eigenvalues in the radial factor of the product solution yields

$$r^2 R''(r) + r R'(r) - 4n^2 R(r) = 0$$

$$R_n(r) = A_n r^{-2n} + B_n r^{2n}.$$

## Product Solution

Define the product solution

$$u_n(r, \theta) = R_n(r) T_n(\theta) = (A_n r^{-2n} + B_n r^{2n}) \sin(2n\theta).$$

By the Principle of Superposition a linear combination of product solutions will also solve Laplace's equation and satisfy the boundary conditions at  $\theta = 0$  and  $\theta = \pi/2$ .

$$u(r, \theta) = \sum_{n=1}^{\infty} (A_n r^{-2n} + B_n r^{2n}) \sin(2n\theta)$$

Now we must choose the coefficients  $A_n$  and  $B_n$  so that the boundary conditions at  $r = 1$  and  $r = 2$  are satisfied. Fourier series will be employed.

## Boundary Condition at $r = 1$

$$u(1, \theta) = \sum_{n=1}^{\infty} (A_n + B_n) \sin(2n\theta)$$
$$\sin(2\theta) = \sum_{n=1}^{\infty} (A_n + B_n) \sin(2n\theta)$$

The last equation implies the system of equations

$$A_1 + B_1 = 1$$
$$A_n + B_n = 0 \text{ for } n \geq 2.$$

## Boundary Condition at $r = 2$

$$u(2, \theta) = \sum_{n=1}^{\infty} (A_n 2^{-2n} + B_n 2^{2n}) \sin(2n\theta)$$
$$\theta \left( \frac{\pi}{2} - \theta \right) = \sum_{n=1}^{\infty} (A_n 2^{-2n} + B_n 2^{2n}) \sin(2n\theta)$$

Multiply both sides of the last equation by  $\sin(2m\theta)$  and integrate over  $[0, \pi/2]$ .

$$\int_0^{\pi/2} \theta \left( \frac{\pi}{2} - \theta \right) \sin(2m\theta) d\theta = \sum_{n=1}^{\infty} (A_n 2^{-2n} + B_n 2^{2n}) \int_0^{\pi/2} \sin(2n\theta) \sin(2m\theta) d\theta$$
$$\frac{1 - (-1)^m}{4m^3} = \frac{\pi/2}{2} (A_m 2^{-2m} + B_m 2^{2m})$$
$$\frac{1 - (-1)^m}{\pi m^3} = A_m 2^{-2m} + B_m 2^{2m}$$

## Systems of Equations (1 of 3)

$$A_1 + B_1 = 1$$

$$\frac{1}{4}A_1 + 4B_1 = \frac{2}{\pi}$$

which implies  $A_1 = \frac{16\pi - 8}{15\pi}$  and  $B_1 = \frac{8 - \pi}{15\pi}$ .

## Systems of Equations (2 of 3)

For  $n = (2k + 1)$  (*i.e.*,  $n$  odd and greater than 1)

$$A_{2k+1} + B_{2k+1} = 0$$

$$2^{-4k-2}A_{2k+1} + 2^{4k+2}B_{2k+1} = \frac{2}{\pi(2k+1)^3}$$

which implies

$$A_{2k+1} = \frac{2^{4k+3}}{\pi(2k+1)^3(2^{4(2k+1)} - 1)}$$

$$B_{2k+1} = -\frac{2^{4k+3}}{\pi(2k+1)^3(2^{4(2k+1)} - 1)}.$$

# Systems of Equations (3 of 3)

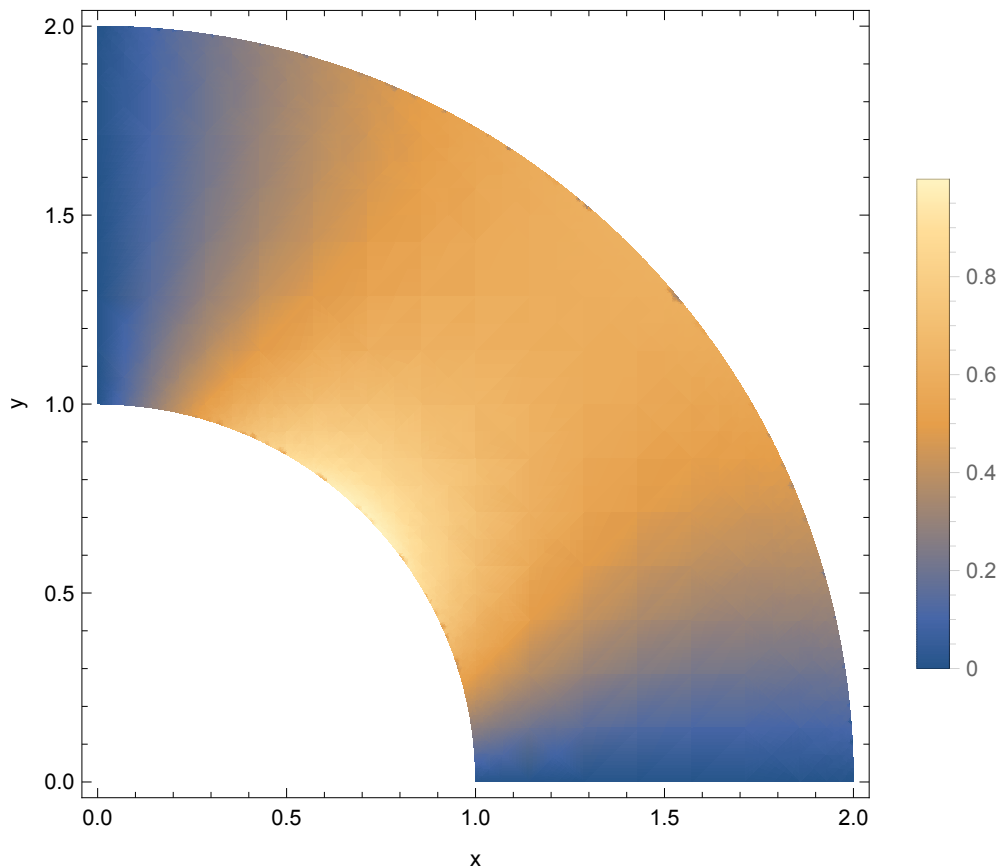
For  $n = 2k$  ( $n$  even)

$$\begin{aligned}A_{2k} + B_{2k} &= 0 \\ 2^{-4k}A_{2k} + 2^{4k}B_{2k} &= 0\end{aligned}$$

which implies  $A_{2k} = B_{2k} = 0$ .

$$\begin{aligned}u(r, \theta) = & \left( \frac{(16\pi - 8)r^{-2}}{15\pi} + \frac{(8 - \pi)r^2}{15\pi} \right) \sin(2\theta) \\ & + \sum_{n=1}^{\infty} \frac{2^{4k+3}(r^{-4k-2} - r^{4k+2})}{\pi(2k+1)^3(2^{4(2k+1)} - 1)} \sin((4n+2)\theta)\end{aligned}$$

## Illustration



# Homework

- ▶ Read Section 6.3–6.4
- ▶ Exercises: 10–15