

Laplace's Equation on a Disk

Consider the Dirichlet boundary value problem:

$$\Delta u = 0 \text{ for } x^2 + y^2 < a^2$$

$$u(x, y) = \phi(x, y) \text{ for } x^2 + y^2 = a^2.$$

Remark: since the boundary of Ω is not a rectangle, we cannot use separation of variables directly. Instead we must convert to polar coordinates before using separation of variables.

In polar coordinates (r, θ) the Laplacian operator can be expressed as

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta},$$

for $r > 0$.

Dirichlet BVP in Polar Coordinates

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \text{ for } 0 < r < a \text{ and } -\infty < \theta < \infty$$

$$u(a, \theta) = f(\theta) = \phi(a \cos \theta, a \sin \theta)$$

Remarks:

- ▶ The boundary conditions are periodic in θ with period 2π .
- ▶ The solution $u(r, \theta)$ should be 2π -periodic in θ .
- ▶ The Laplacian is not defined for $r = 0$, but we wish for the solution to remain finite as $r \rightarrow 0^+$.

Separation of Variables in Polar Coordinates

Assume $u(r, \theta) = R(r)T(\theta)$, then

$$R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) + \frac{1}{r^2}R(r)T''(\theta) = 0$$

$$\frac{R''(r)T(\theta)}{R(r)T(\theta)} + \frac{1}{r}\frac{R'(r)T(\theta)}{R(r)T(\theta)} + \frac{1}{r^2}\frac{R(r)T''(\theta)}{R(r)T(\theta)} = 0$$

$$\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} = c$$

where c is a constant.

<div data-bbox="23 1435 67 2150" data-label="Section-Header"> <h2>Implied Ordinary Differential Equations</h2> </div> <div data-bbox="215 1525 250 1767" data-label="Equation-Block"> $T''(\theta) + c T(\theta) = 0$ </div> <div data-bbox="314 1397 346 2076" data-label="Text"> <p>case $c = 0$: the only nontrivial 2π-periodic solution is</p> </div> <div data-bbox="376 1489 410 1626" data-label="Equation-Block"> $T_0(\theta) = A_0$ </div> <div data-bbox="440 1344 474 2094" data-label="Text"> <p>case $c = \lambda^2 > 0$: the only nontrivial 2π-periodic solution is</p> </div> <div data-bbox="504 1361 539 1753" data-label="Equation-Block"> $T_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$ </div> <div data-bbox="568 1574 603 1912" data-label="Text"> <p>and $c = \lambda_n^2 = n^2$ for $n \in \mathbb{N}$.</p> </div>	<div data-bbox="23 795 67 1095" data-label="Section-Header"> <h2>Euler's Equation</h2> </div> <div data-bbox="119 378 151 1037" data-label="Text"> <p>The implied ordinary differential equation for $R(r)$ is</p> </div> <div data-bbox="181 396 217 790" data-label="Equation-Block"> $r^2 R''(r) + r R'(r) - c R(r) = 0,$ </div> <div data-bbox="247 566 279 1037" data-label="Text"> <p>which is known as Euler's equation.</p> </div> <div data-bbox="293 871 320 1021" data-label="Text"> <p>case $c = 0$:</p> </div> <div data-bbox="351 340 430 667" data-label="Equation-Block"> $\begin{aligned} 0 &= r^2 R''(r) + r R'(r) \\ R_0(r) &= C_0 \ln r + D_0 \end{aligned}$ </div> <div data-bbox="462 822 496 1037" data-label="Text"> <p>case $c = n^2 > 0$:</p> </div> <div data-bbox="526 277 608 730" data-label="Equation-Block"> $\begin{aligned} 0 &= r^2 R''(r) + r R'(r) - n^2 R(r) \\ R_n(r) &= C_n r^{-n} + D_n r^n \end{aligned}$ </div> <div data-bbox="639 188 707 1037" data-label="Text"> <p>In order for the product solution to be bounded as $r \rightarrow 0^+$ we must choose $C_0 = C_1 = \dots = 0$.</p> </div>
<div data-bbox="821 1843 865 2150" data-label="Section-Header"> <h2>Product Solution</h2> </div> <div data-bbox="948 1850 975 2092" data-label="Text"> <p>Define the function</p> </div> <div data-bbox="1010 1328 1043 1968" data-label="Equation-Block"> $u_n(r, \theta) = R_n(r) T_n(\theta) = r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$ </div> <div data-bbox="1075 1897 1106 2092" data-label="Text"> <p>for $n = 0, 1, \dots$</p> </div> <div data-bbox="1131 1514 1163 2092" data-label="Text"> <p>By the Principle of Superposition the function</p> </div> <div data-bbox="1193 1366 1279 1933" data-label="Equation-Block"> $u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$ </div> <div data-bbox="1310 1211 1375 2092" data-label="Text"> <p>satisfies the Laplacian on the disk for any choice of constants A_0, A_n, and B_n.</p> </div> <div data-bbox="1401 1301 1465 2092" data-label="Text"> <p>Fourier series techniques can be used to satisfy the boundary condition.</p> </div>	<div data-bbox="821 728 865 1095" data-label="Section-Header"> <h2>Boundary Condition</h2> </div> <div data-bbox="928 721 960 1037" data-label="Text"> <p>Since $u(a, \theta) = f(\theta)$ then</p> </div> <div data-bbox="991 318 1077 860" data-label="Equation-Block"> $f(\theta) = A_0 + \sum_{n=1}^{\infty} a^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$ </div> <div data-bbox="1091 508 1157 844" data-label="Equation-Block"> $A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta = \frac{\alpha_0}{2}$ </div> <div data-bbox="1171 432 1236 871" data-label="Equation-Block"> $a^n A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta = \alpha_n$ </div> <div data-bbox="1251 436 1316 871" data-label="Equation-Block"> $a^n B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta = \beta_n$ </div> <div data-bbox="1355 987 1380 1037" data-label="Text"> <p>and</p> </div> <div data-bbox="1380 280 1465 907" data-label="Equation-Block"> $u(r, \theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n [\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)].$ </div>

Example

Solve Laplace’s equation on the unit disk with the following Dirichlet boundary condition.

$$\Delta u = 0 \text{ for } x^2 + y^2 < 1$$
$$u(1, \theta) = \pi - \theta \text{ for } -\pi < \theta < \pi$$

Solution

The solution takes the form

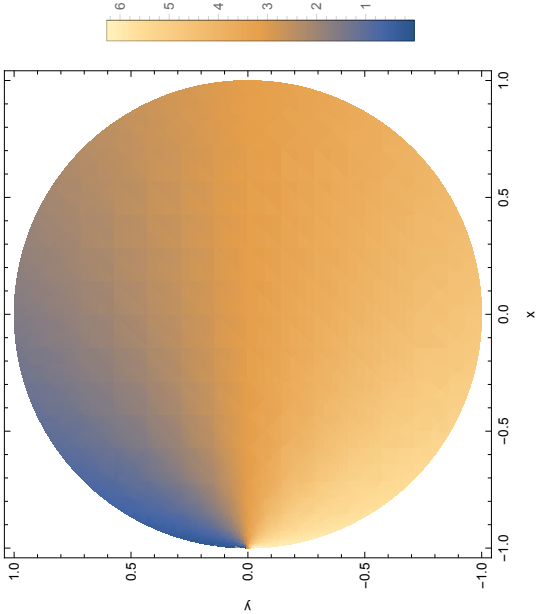
$$u(r, \theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n [\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)]$$

where

$$\alpha_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - \theta) \, d\theta = 2\pi$$
$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - \theta) \cos(n\theta) \, d\theta = 0$$
$$\beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - \theta) \sin(n\theta) \, d\theta = \frac{2(-1)^n}{n}.$$
$$u(r, \theta) = \pi + \sum_{n=1}^{\infty} \frac{2(-1)^n r^n}{n} \sin(n\theta)$$

Illustration

$$u(r, \theta) = \pi + \sum_{n=1}^{\infty} \frac{2(-1)^n r^n}{n} \sin(n\theta)$$



Complex Arithmetic (1 of 3)

Suppose $z = a + ib$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

$$\sum_{n=0}^{\infty} (-z)^n = \frac{1}{1+z} \text{ (if } |z| < 1)$$
$$\sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1} = \ln(1+z)$$
$$-\sum_{n=1}^{\infty} \frac{(-z)^n}{n} = \ln(1+z)$$

Complex Arithmetic (2 of 3)

In polar coordinate form $z = r e^{i\theta}$ where $r = \sqrt{a^2 + b^2} = |z|$ and $\theta = \tan^{-1}(b/a)$.

$$-\sum_{n=1}^{\infty} \frac{(-z)^n}{n} = \ln(1+z)$$
$$-\sum_{n=1}^{\infty} \frac{1}{n} (-re^{i\theta})^n = \ln(1+re^{i\theta}) \text{ (if } |r| < 1)$$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n e^{in\theta} = \ln(1+re^{i\theta})$$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \operatorname{Im}(e^{in\theta}) = \operatorname{Im}(\ln(1+re^{i\theta}))$$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta) = \operatorname{Im}(\ln(1+r\cos\theta + ir\sin\theta))$$

Complex Arithmetic (3 of 3)

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta) = \operatorname{Im}(\ln(1+r\cos\theta + ir\sin\theta))$$
$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta) = \tan^{-1}\left(\frac{r\sin\theta}{1+r\cos\theta}\right)$$

$$u(r,\theta) = \pi + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta)$$
$$= \pi - 2 \tan^{-1}\left(\frac{r\sin\theta}{1+r\cos\theta}\right)$$

Natural Logarithm of a Complex Number

Suppose $z = a + ib \in \mathbb{C}$ and $w = \alpha + i\beta \in \mathbb{C}$ such that

$$e^w = z$$

then $w = \ln z$.

$$e^{\alpha+i\beta} = a + ib$$

$$e^{\alpha} e^{i\beta} = a + ib$$

$$e^{\alpha}(\cos\beta + i\sin\beta) = a + ib$$

Thus $a = e^{\alpha} \cos\beta$ and $b = e^{\alpha} \sin\beta$ and

$$a^2 + b^2 = e^{2\alpha} \implies \alpha = \ln \sqrt{a^2 + b^2}$$

$$\frac{b}{a} = \tan\beta \implies \beta = \tan^{-1} \frac{b}{a}$$

Thus

$$\ln z = \ln \sqrt{a^2 + b^2} + i \tan^{-1} \frac{b}{a}$$

Laplace's Equation on a Sector of an Annulus

Find the solution to the following boundary value problem.

$$\Delta u = 0 \text{ for } 1 < x^2 + y^2 < 4 \text{ with } x > 0 \text{ and } y > 0$$

$$u(x, 0) = 0$$

$$u(0, y) = 0$$

$$u(x, y) = 2xy \text{ for } x^2 + y^2 = 1$$

$$u(x, y) = \left(\frac{\pi}{2} - \tan^{-1} \frac{y}{x}\right) \tan^{-1} \frac{y}{x} \text{ for } x^2 + y^2 = 4$$

Polar Coordinates

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \text{ for } 1 < r < 2 \text{ with } 0 < \theta < \pi/2$$

$$u(r, 0) = 0$$

$$u(r, \pi/2) = 0$$

$$u(1, \theta) = \sin(2\theta)$$

$$u(2, \theta) = \theta \left(\frac{\pi}{2} - \theta \right)$$

We will again use separation of variables.

Separation of Variables

Assuming $u(r, \theta) = R(r)T(\theta)$ then

$$R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) + \frac{1}{r^2}R(r)T''(\theta) = 0$$

$$\frac{r^2 R''(r)}{R(r)} + \frac{r R'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} = c.$$

This implies the following boundary value problem for the angular factor of the solution.

$$T''(\theta) + cT(\theta) = 0$$

$$T(0) = 0$$

$$T(\pi/2) = 0$$

Eigenfunctions and Eigenvalues

The only nontrivial solutions to

$$T''(\theta) + cT(\theta) = 0$$

$$T(0) = 0$$

$$T(\pi/2) = 0$$

are

$$T_n(\theta) = \sin(2n\theta)$$

with $c = \lambda_n^2 = 4n^2$ for $n \in \mathbb{N}$.

Using the eigenvalues in the radial factor of the product solution yields

$$r^2 R''(r) + r R'(r) - 4n^2 R(r) = 0$$

$$R_n(r) = A_n r^{-2n} + B_n r^{2n}.$$

Product Solution

Define the product solution

$$u_n(r, \theta) = R_n(r)T_n(\theta) = (A_n r^{-2n} + B_n r^{2n}) \sin(2n\theta).$$

By the Principle of Superposition a linear combination of product solutions will also solve Laplace's equation and satisfy the boundary conditions at $\theta = 0$ and $\theta = \pi/2$.

$$u(r, \theta) = \sum_{n=1}^{\infty} (A_n r^{-2n} + B_n r^{2n}) \sin(2n\theta)$$

Now we must choose the coefficients A_n and B_n so that the boundary conditions at $r = 1$ and $r = 2$ are satisfied. Fourier series will be employed.

Boundary Condition at $r = 1$

$$u(1, \theta) = \sum_{n=1}^{\infty} (A_n + B_n) \sin(2n\theta)$$

$$\sin(2\theta) = \sum_{n=1}^{\infty} (A_n + B_n) \sin(2n\theta)$$

The last equation implies the system of equations

$$A_1 + B_1 = 1$$

$$A_n + B_n = 0 \text{ for } n \geq 2.$$

Boundary Condition at $r = 2$

$$u(2, \theta) = \sum_{n=1}^{\infty} (A_n 2^{-2n} + B_n 2^{2n}) \sin(2n\theta)$$

$$\theta \left(\frac{\pi}{2} - \theta \right) = \sum_{n=1}^{\infty} (A_n 2^{-2n} + B_n 2^{2n}) \sin(2n\theta)$$

Multiply both sides of the last equation by $\sin(2m\theta)$ and integrate over $[0, \pi/2]$.

$$\int_0^{\pi/2} \theta \left(\frac{\pi}{2} - \theta \right) \sin(2m\theta) d\theta = \sum_{n=1}^{\infty} (A_n 2^{-2n} + B_n 2^{2n}) \int_0^{\pi/2} \sin(2n\theta) \sin(2m\theta) d\theta$$

$$\frac{1 - (-1)^m}{4m^3} = \frac{\pi/2}{2} (A_m 2^{-2m} + B_m 2^{2m})$$

$$\frac{1 - (-1)^m}{\pi m^3} = A_m 2^{-2m} + B_m 2^{2m}$$

Systems of Equations (1 of 3)

$$A_1 + B_1 = 1$$

$$\frac{1}{4} A_1 + 4B_1 = \frac{2}{\pi}$$

which implies $A_1 = \frac{16\pi - 8}{15\pi}$ and $B_1 = \frac{8 - \pi}{15\pi}$.

Systems of Equations (2 of 3)

For $n = (2k + 1)$ (i.e., n odd and greater than 1)

$$A_{2k+1} + B_{2k+1} = 0$$

$$2^{-4k-2} A_{2k+1} + 2^{4k+2} B_{2k+1} = \frac{2}{\pi(2k+1)^3}$$

which implies

$$A_{2k+1} = \frac{2^{4k+3}}{\pi(2k+1)^3(2^{4(2k+1)} - 1)}$$

$$B_{2k+1} = -\frac{2^{4k+3}}{\pi(2k+1)^3(2^{4(2k+1)} - 1)}.$$

Systems of Equations (3 of 3)

For $n = 2k$ (n even)

$$\begin{aligned} A_{2k} + B_{2k} &= 0 \\ 2^{-4k} A_{2k} + 2^{4k} B_{2k} &= 0 \end{aligned}$$

which implies $A_{2k} = B_{2k} = 0$.

$$\begin{aligned} u(r, \theta) = & \left(\frac{(16\pi - 8)r^{-2}}{15\pi} + \frac{(8 - \pi)r^2}{15\pi} \right) \sin(2\theta) \\ & + \sum_{n=1}^{\infty} \frac{2^{4k+3}(r^{-4k-2} - r^{4k+2})}{\pi(2k+1)^3(2^{4(2k+1)} - 1)} \sin((4n+2)\theta) \end{aligned}$$

Homework

- ▶ Read Section 6.3–6.4
- ▶ Exercises: 10–15

Illustration

