Laplace's Equation on a Disk

Partial Differential Equations

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Consider the Dirichlet boundary value problem:

$$\Delta u = 0 \text{ for } x^2 + y^2 < a^2$$
$$u(x, y) = \phi(x, y) \text{ for } x^2 + y^2 = a^2.$$

Remark: since the boundary of Ω is not a rectangle, we cannot use separation of variables directly. Instead we must convert to polar coordinates before using separation of variables.

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Remark: since the boundary of Ω is not a rectangle, we cannot use separation of variables directly. Instead we must convert to polar coordinates before using separation of variables.

In polar coordinates (r, θ) the Laplacian operator can be expressed as

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta},$$

for $r > 0$.

Dirichlet BVP in Polar Coordinates

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \text{ for } 0 < r < a \text{ and } -\infty < \theta < \infty$$
$$u(a,\theta) = f(\theta) = \phi(a\cos\theta, a\sin\theta)$$

Remarks:

- The boundary conditions are periodic in θ with period 2π.
- The solution $u(r, theta)$ should be 2π-periodic in θ.
- The Laplacian is not defined for $r = 0$, but we wish for the solution to remain finite as $r \to 0^+$.

Separation of Variables in Polar Coordinates

Assume $u(r,\theta) = R(r)T(\theta)$, then

$$
\begin{aligned}
R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) + \frac{1}{r^2}R(r)T''(\theta) &= 0 \\
r^2\left(\frac{R''(r)T(\theta)}{R(r)T(\theta)} + \frac{1}{r}\frac{R'(r)T(\theta)}{R(r)T(\theta)} + \frac{1}{r^2}\frac{R(r)T''(\theta)}{R(r)T(\theta)}\right) &= 0\cdot r^2 \\
\frac{r^2 R''(r)}{R(r)} + \frac{r\,R'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} &= c
\end{aligned}
$$

where c is a constant.

Implied Ordinary Differential Equations

$$T''(\theta) + c\,T(\theta) = 0$$

case $c = 0$: the only nontrivial 2π-periodic solution is

$$T_0(\theta) = A_0$$

case $c = \lambda^2 > 0$: the only nontrivial 2π-periodic solution is

$$T_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$$

and $c = \lambda_n^2 = n^2$ for $n \in \mathbb{N}$.

Euler's Equation

The implied ordinary differential equation for $R(r)$ is

$$r^2 R''(r) + r\,R'(r) - c\,R(r) = 0,$$

which is known as **Euler's equation**.

case $c = 0$:

$$\begin{aligned} 0 &= r^2 R''(r) + r\,R'(r) \\ R_0(r) &= C_0 \ln r + D_0 \end{aligned}$$

case $c = n^2 > 0$:

$$\begin{aligned} 0 &= r^2 R''(r) + r\,R'(r) - n^2\,R(r) \\ R_n(r) &= C_n r^{-n} + D_n r^n \end{aligned}$$

In order for the product solution to be bounded as $r \to 0^+$ we must choose $C_0 = C_1 = \cdots = 0$.

Product Solution

Define the function

$$u_n(r, \theta) = R_n(r) T_n(\theta) = r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta) \right]$$

for $n = 0, 1, \ldots$.

By the Principle of Superposition the function

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta) \right]$$

satisfies the Laplacian on the disk for any choice of constants A_0, A_n, and B_n.

Fourier series techniques can be used to satisfy the boundary condition.

Boundary Condition

Since $u(a, \theta) = f(\theta)$ then

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} a^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$$

Boundary Condition

Since $u(a, \theta) = f(\theta)$ then

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} a^n \left[A_n \cos(n\theta) + B_n \sin(n\theta) \right]$$

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta = \frac{\alpha_0}{2}$$

$$a^n A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta = \alpha_n$$

$$a^n B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta = \beta_n$$

Boundary Condition

Since $u(a,\theta) = f(\theta)$ then

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$$a^n B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta = \beta_n$$

and

$$u(r,\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)\right].$$

Example

Solve Laplace's equation on the unit disk with the following Dirichlet boundary condition.

$$\Delta u = 0 \text{ for } x^2 + y^2 < 1$$
$$u(1, \theta) = \pi - \theta \text{ for } -\pi < \theta < \pi$$

Solution

The solution takes the form

$$u(r,\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n \left[\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)\right]$$

where

$$\begin{aligned}
\alpha_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - \theta)\, d\theta = 2\pi \\
\alpha_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - \theta)\cos(n\theta)\, d\theta = 0 \\
\beta_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - \theta)\sin(n\theta)\, d\theta = \frac{2(-1)^n}{n}.
\end{aligned}$$

Solution

The solution takes the form

$$u(r,\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n \left[\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)\right]$$

where

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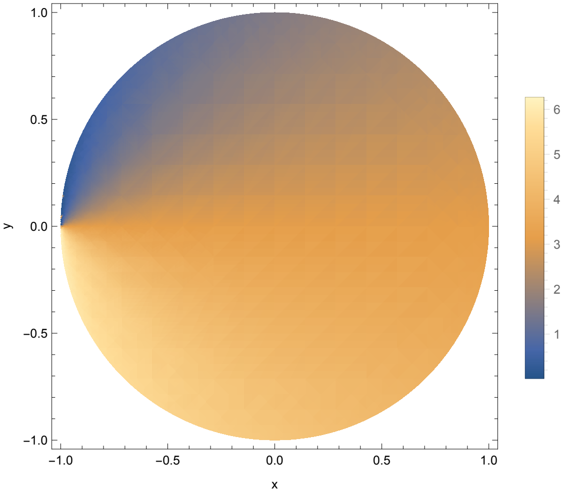
\beta_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - \theta)\sin(n\theta)\, d\theta = \frac{2(-1)^n}{n}.
\end{aligned}$$

$$u(r,\theta) = \pi + \sum_{n=1}^{\infty} \frac{2(-1)^n r^n}{n} \sin(n\theta)$$

Illustration

$$u(r,\theta) = \pi + \sum_{n=1}^{\infty} \frac{2(-1)^n r^n}{n} \sin(n\theta)$$



Summing the Series

It is possible to sum the Fourier series by recalling the geometric series and using some complex number arithmetic.

- Geometric Series: $\displaystyle\sum_{n=0}^{\infty} a\,r^n = \frac{a}{1-r}$ if $|r| < 1$.
- Euler's Identity: $e^{i\,\theta} = \cos\theta + i\,\sin\theta$.
- If $z = a + i\,b$ then $|z| = \sqrt{z\,\overline{z}} = \sqrt{a^2 + b^2}$.

Complex Arithmetic (1 of 3)

Suppose $z = a + i\,b$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. The geometric series in z is

$$\sum_{n=0}^{\infty} (-z)^n = \frac{1}{1+z} \text{ (if } |z| < 1).$$

Integrate both sides of this equation with respect to z.

$$\sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1} = \ln(1+z) \text{ (and re-index)}$$

$$-\sum_{n=1}^{\infty} \frac{(-z)^n}{n} = \ln(1+z)$$

Complex Arithmetic (2 of 3)

In polar coordinate form $z = r\,e^{i\,\theta}$ where $r = \sqrt{a^2 + b^2} = |z|$ and $\theta = \tan^{-1}(b/a)$.

$$-\sum_{n=1}^{\infty} \frac{(-z)^n}{n} = \ln(1 + z)$$

Complex Arithmetic (2 of 3)

In polar coordinate form $z = r\,e^{i\,\theta}$ where $r = \sqrt{a^2 + b^2} = |z|$ and $\theta = \tan^{-1}(b/a)$.

$$-\sum_{n=1}^{\infty} \frac{(-z)^n}{n} = \ln(1 + z)$$

$$-\sum_{n=1}^{\infty} \frac{1}{n}(-re^{i\theta})^n = \ln(1 + re^{i\theta}) \text{ (if } |r| < 1\text{)}$$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n e^{in\theta} = \ln(1 + re^{i\theta}) \text{ (find imaginary part)}$$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \operatorname{Im}\left(e^{in\theta}\right) = \operatorname{Im}\left(\ln(1 + re^{i\theta})\right)$$

Complex Arithmetic (2 of 3)

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\begin{aligned}
-\sum_{n=1}^{\infty} \frac{(-z)^n}{n} &= \ln(1+z) \\
-\sum_{n=1}^{\infty} \frac{1}{n}(-re^{i\theta})^n &= \ln(1+re^{i\theta}) \text{ (if } |r| < 1\text{)} \\
-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n e^{in\theta} &= \ln(1+re^{i\theta}) \text{ (find imaginary part)} \\
-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \operatorname{Im}\left(e^{in\theta}\right) &= \operatorname{Im}\left(\ln(1+re^{i\theta})\right) \\
-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta) &= \operatorname{Im}\left(\ln(1 + r\cos\theta + ir\sin\theta)\right)
\end{aligned}
$$

Complex Arithmetic (2 of 3)

In polar coordinate form $z = r\,e^{i\,\theta}$ where $r = \sqrt{a^2 + b^2} = |z|$ and $\theta = \tan^{-1}(b/a)$.

$$-\sum_{n=1}^{\infty} \frac{(-z)^n}{n} = \ln(1+z)$$

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$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta) = \operatorname{Im}\left(\ln(1 + r\cos\theta + ir\sin\theta)\right)$$

Remark: we must determine the natural logarithm of a complex number so that we may find its imaginary part.

Natural Logarithm of a Complex Number

Suppose $z = a + ib \in \mathbb{C}$ and $w = \alpha + i\beta \in \mathbb{C}$ such that

$$e^w = z$$

then $w = \ln z$.

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then $w = \ln z$.

$$\begin{aligned} e^{\alpha+i\beta} &= a + ib \\ e^{\alpha} e^{i\beta} &= a + ib \\ e^{\alpha}(\cos\beta + i\sin\beta) &= a + ib \end{aligned}$$

Thus $a = e^{\alpha}\cos\beta$ and $b = e^{\alpha}\sin\beta$ and

$$a^2 + b^2 = e^{2\alpha} \implies \alpha = \ln\sqrt{a^2 + b^2}$$

$$\frac{b}{a} = \tan\beta \implies \beta = \tan^{-1}\frac{b}{a}.$$

Thus

$$\ln z = \ln\sqrt{a^2 + b^2} + i\tan^{-1}\frac{b}{a}.$$

Complex Arithmetic (3 of 3)

Returning to a previous equation:

$$
\begin{aligned}
-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta) &= \operatorname{Im}\left(\ln(1 + r\cos\theta + ir\sin\theta)\right) \\
-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta) &= \tan^{-1}\left(\frac{r\sin\theta}{1 + r\cos\theta}\right) \\
2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta) &= -2\tan^{-1}\left(\frac{r\sin\theta}{1 + r\cos\theta}\right) \\
u(r,\theta) = \pi + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n \sin(n\theta) &= \pi - 2\tan^{-1}\left(\frac{r\sin\theta}{1 + r\cos\theta}\right).
\end{aligned}
$$

Thus we have summed the infinite series.

Laplace's Equation on a Sector of an Annulus

Find the solution to the following boundary value problem.

$$\Delta u = 0 \text{ for } 1 < x^2 + y^2 < 4 \text{ with } x > 0 \text{ and } y > 0$$
$$u(x, 0) = 0$$
$$u(0, y) = 0$$
$$u(x, y) = 2xy \text{ for } x^2 + y^2 = 1$$
$$u(x, y) = \left(\frac{\pi}{2} - \tan^{-1}\frac{y}{x}\right)\tan^{-1}\frac{y}{x} \text{ for } x^2 + y^2 = 4$$

Polar Coordinates

$$
\begin{aligned}
u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} &= 0 \text{ for } 1 < r < 2 \text{ with } 0 < \theta < \pi/2 \\
u(r, 0) &= 0 \\
u(r, \pi/2) &= 0 \\
u(1, \theta) &= \sin(2\theta) \\
u(2, \theta) &= \theta \left(\frac{\pi}{2} - \theta \right)
\end{aligned}
$$

We will again use separation of variables.

Separation of Variables

Assuming $u(r,\theta) = R(r)T(\theta)$ then

$$R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) + \frac{1}{r^2}R(r)T''(\theta) = 0$$
$$\frac{r^2 R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} = c.$$

This implies the following boundary value problem for the angular factor of the solution.

$$T''(\theta) + cT(\theta) = 0$$
$$T(0) = 0$$
$$T(\pi/2) = 0$$

Eigenfunctions and Eigenvalues

The only nontrivial solutions to

$$\begin{aligned} T''(\theta) + cT(\theta) &= 0 \\ T(0) &= 0 \\ T(\pi/2) &= 0 \end{aligned}$$

are

$$T_n(\theta) = \sin(2n\theta)$$

with $c = \lambda_n^2 = 4n^2$ for $n \in \mathbb{N}$.

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are

$$T_n(\theta) = \sin(2n\theta)$$

with $c = \lambda_n^2 = 4n^2$ for $n \in \mathbb{N}$.

Using the eigenvalues in the radial factor of the product solution yields

$$\begin{aligned} r^2 R''(r) + r\,R'(r) - 4n^2 R(r) &= 0 \\ R_n(r) &= A_n r^{-2n} + B_n r^{2n}. \end{aligned}$$

Product Solution

Define the product solution

$$u_n(r,\theta) = R_n(r)T_n(\theta) = \left(A_n r^{-2n} + B_n r^{2n}\right)\sin(2n\theta).$$

By the Principle of Superposition a linear combination of product solutions will also solve Laplace's equation and satisfy the boundary conditions at $\theta = 0$ and $\theta = \pi/2$.

$$u(r,\theta) = \sum_{n=1}^{\infty} \left(A_n r^{-2n} + B_n r^{2n}\right)\sin(2n\theta)$$

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$$u(r,\theta) = \sum_{n=1}^{\infty} \left(A_n r^{-2n} + B_n r^{2n}\right)\sin(2n\theta)$$

Now we must choose the coefficients A_n and B_n so that the boundary conditions at $r = 1$ and $r = 2$ are satisfied. Fourier series will be employed.

Boundary Condition at $r = 1$

$$u(1,\theta) = \sum_{n=1}^{\infty} (A_n + B_n)\sin(2n\theta)$$

$$\sin(2\theta) = \sum_{n=1}^{\infty} (A_n + B_n)\sin(2n\theta)$$

The last equation implies the system of equations

$$\begin{aligned} A_1 + B_1 &= 1 \\ A_n + B_n &= 0 \text{ for } n \geq 2. \end{aligned}$$

Boundary Condition at $r = 2$

$$u(2,\theta) = \sum_{n=1}^{\infty} \left(A_n 2^{-2n} + B_n 2^{2n} \right) \sin(2n\theta)$$

$$\theta \left(\frac{\pi}{2} - \theta \right) = \sum_{n=1}^{\infty} \left(A_n 2^{-2n} + B_n 2^{2n} \right) \sin(2n\theta)$$

Multiply both sides of the last equation by $\sin(2m\theta)$ and integrate over $[0, \pi/2]$.

$$\int_0^{\pi/2} \theta \left(\frac{\pi}{2} - \theta \right) \sin(2m\theta)\, d\theta = \sum_{n=1}^{\infty} \left(A_n 2^{-2n} + B_n 2^{2n} \right) \int_0^{\pi/2} \sin(2n\theta) \sin(2m\theta)\, d\theta$$

$$\frac{1-(-1)^m}{4m^3} = \frac{\pi/2}{2} \left(A_m 2^{-2m} + B_m 2^{2m} \right)$$

$$\frac{1-(-1)^m}{\pi m^3} = A_m 2^{-2m} + B_m 2^{2m}$$

Systems of Equations (1 of 3)

$$\begin{aligned} A_1 + B_1 &= 1 \\ \frac{1}{4}A_1 + 4B_1 &= \frac{2}{\pi} \end{aligned}$$

which implies $A_1 = \dfrac{16\pi - 8}{15\pi}$ and $B_1 = \dfrac{8 - \pi}{15\pi}$.

Systems of Equations (2 of 3)

For $n = (2k+1)$ (*i.e.*, n odd and greater than 1)

$$\begin{aligned} A_{2k+1} + B_{2k+1} &= 0 \\ 2^{-4k-2} A_{2k+1} + 2^{4k+2} B_{2k+1} &= \frac{2}{\pi(2k+1)^3} \end{aligned}$$

which implies

$$\begin{aligned} A_{2k+1} &= \frac{2^{4k+3}}{\pi(2k+1)^3(2^{4(2k+1)}-1)} \\ B_{2k+1} &= -\frac{2^{4k+3}}{\pi(2k+1)^3(2^{4(2k+1)}-1)}. \end{aligned}$$

Systems of Equations (3 of 3)

For $n = 2k$ (n even)

$$\begin{aligned} A_{2k} + B_{2k} &= 0 \\ 2^{-4k} A_{2k} + 2^{4k} B_{2k} &= 0 \end{aligned}$$

which implies $A_{2k} = B_{2k} = 0$.

Systems of Equations (3 of 3)

For $n = 2k$ (n even)

$$\begin{aligned} A_{2k} + B_{2k} &= 0 \\ 2^{-4k} A_{2k} + 2^{4k} B_{2k} &= 0 \end{aligned}$$

which implies $A_{2k} = B_{2k} = 0$.

$$\begin{aligned} u(r,\theta) = &\left(\frac{(16\pi - 8)r^{-2}}{15\pi} + \frac{(8-\pi)r^2}{15\pi} \right) \sin(2\theta) \\ &+ \sum_{n=1}^{\infty} \frac{2^{4k+3}(r^{-4k-2} - r^{4k+2})}{\pi(2k+1)^3(2^{4(2k+1)}-1)} \sin((4n+2)\theta) \end{aligned}$$

Illustration