Mixed Boundary Conditions on Rectangles

Partial Differential Equations

J Robert Buchanan

Department of Mathematics

Fall 2025

Objectives

In this lesson we will learn to:

- solve Laplace's equation on a rectangle with mixed Dirichlet and Neumann boundary conditions,
- periodically extend boundary conditions to obtain Fourier series solution containing just even-indexed or just odd-indexed terms.

General Boundary Value Problem

Consider the following mixed boundary value problem on a rectangle in the xy-plane:

$$\begin{aligned}
\Delta u &= 0 \text{ for } 0 < x < a \text{ and } 0 < y < b \\
u(x,0) &= 0 \text{ for } 0 < x < a \\
u_y(x,b) &= 0 \text{ for } 0 < x < a \\
u(0,y) &= 0 \text{ for } 0 < y < b \\
u(a,y) &= f(y) \text{ for } 0 < y < b.
\end{aligned}$$

Physical Interpretation: the left and bottom edges of the rectangle are kept at temperature 0, the top edge is insulated, and along the right edge the distribution of temperature has been specified as $f(y)$.

Separation of Variables (1 of 2)

Assume a product solution $u(x, y) = X(x)Y(y)$, then the following ODEs and BCs are induced:

$$X''(x) - \sigma X(x) = 0 \text{ for } 0 < x < a$$
$$X(0) = 0,$$

and

$$Y''(y) + \sigma Y(y) = 0 \text{ for } 0 < y < b$$
$$Y(0) = Y'(b) = 0,$$

where σ is a constant.
The only non-trivial solutions for $Y(y)$ occur when $\sigma = \sigma_n = ((2n - 1)\pi/(2b))^2$ and

$$Y_n(y) = \sin \frac{(2n-1)\pi y}{2b} \text{ for } n \in \mathbb{N}.$$

Separation of Variables (2 of 2)

Substituting σ_n into the ODE for $X(x)$ produces

$$X_n(x) = \sinh \frac{(2n-1)\pi x}{2b}.$$

The product solutions have the form:

$$u_n(x,y) = \sinh \frac{(2n-1)\pi x}{2b} \sin \frac{(2n-1)\pi y}{2b}$$

Using the Principle of Superposition, the formal solution to the BVP can be expressed as,

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh \frac{(2n-1)\pi x}{2b} \sin \frac{(2n-1)\pi y}{2b}.$$

The nonhomogeneous BC can be used to determine coefficients c_n.

Boundary Condition at $x = a$

$$u(a, y) = \sum_{n=1}^{\infty} c_n \sinh \frac{(2n-1)\pi a}{2b} \sin \frac{(2n-1)\pi y}{2b} = f(y)$$

Remark: In order to determine the coefficients c_n, $f(y)$ must be extended in such a way that the even-indexed Fourier sine coefficients vanish.

Boundary Condition at $x = a$

$$u(a, y) = \sum_{n=1}^{\infty} c_n \sinh \frac{(2n-1)\pi a}{2b} \sin \frac{(2n-1)\pi y}{2b} = f(y)$$

Remark: In order to determine the coefficients c_n, $f(y)$ must be extended in such a way that the even-indexed Fourier sine coefficients vanish.

Extend $f(y)$ to the interval $[-2b, 2b]$ as $f_o(y)$ where

$$f_o(y) = \begin{cases} -f(2b+y) & \text{if } -2b \leq y \leq -b, \\ -f(-y) & \text{if } -b \leq y \leq 0, \\ f(y) & \text{if } 0 \leq y \leq b \\ f(2b-y) & \text{if } b \leq y \leq 2b. \end{cases}$$

Fourier Coefficients

The Fourier sine series representation of $f_o(y)$ is calculated as

$$f_o(y) \sim \sum_{m=1}^{\infty} \beta_m \sin \frac{(2m-1)\pi y}{2b}.$$

Multiply both sides by $\sin \dfrac{(2n-1)\pi y}{2b}$ and integrate over $[0, b]$.

$$\begin{aligned}
\int_0^b f_o(y) \sin \frac{(2n-1)\pi y}{2b}\, dy &= \sum_{m=1}^{\infty} \beta_m \int_0^b \sin \frac{(2m-1)\pi y}{2b} \sin \frac{(2n-1)\pi y}{2b}\, dy \\
&= \beta_n \left(\frac{b}{2}\right) \text{ (orthogonality)} \\
\beta_n &= \frac{2}{b} \int_0^b f(y) \sin \frac{(2n-1)\pi y}{2b}\, dy
\end{aligned}$$

The coefficients c_n of the series representation of $f(y)$ should be chosen as

$$c_n = \frac{2}{b \sinh \frac{(2n-1)\pi a}{2b}} \int_0^b f(y) \sin \frac{(2n-1)\pi y}{2b}\, dy.$$

Example

Find the solution to Laplace's equation with mixed boundary conditions below. Interpret this boundary value problem in the context of a steady-state solution to the heat equation on a rectangular plate.

$$\begin{aligned}
\Delta u &= 0 \text{ for } 0 < x < 2 \text{ and } 0 < y < 1 \\
u(0, y) &= 0 \text{ for } 0 < y < 1 \\
u(2, y) &= y(1 - y) \text{ for } 0 < y < 1 \\
u(x, 0) &= u_y(x, 1) = 0 \text{ for } 0 < x < 2
\end{aligned}$$

Solution (1 of 4)

Physical Interpretation: the domain $\Omega = \{(x, y) \,|\, 0 < x < 2,\ 0 < y < 1\}$ is a rectangle whose left and bottom edges are kept at constant temperature 0 (perhaps by being in perfect thermal contact with ice). The top edge is insulated so that no heat flows out the top of the rectangle. The right edge of the rectangle is in perfect thermal contact with a heat source with temperature distribution given by $f(y) = y(1 - y)$.

Solution (2 of 4)

As derived previously, the formal solution has the form,

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh \frac{(2n-1)\pi x}{2} \sin \frac{(2n-1)\pi y}{2},$$

since $b = 1$.

At the boundary where $a = 2$,

$$u(2,y) = \sum_{n=1}^{\infty} c_n \sinh((2n-1)\pi) \sin \frac{(2n-1)\pi y}{2} = y(1-y).$$

The coefficients c_n are calculated as

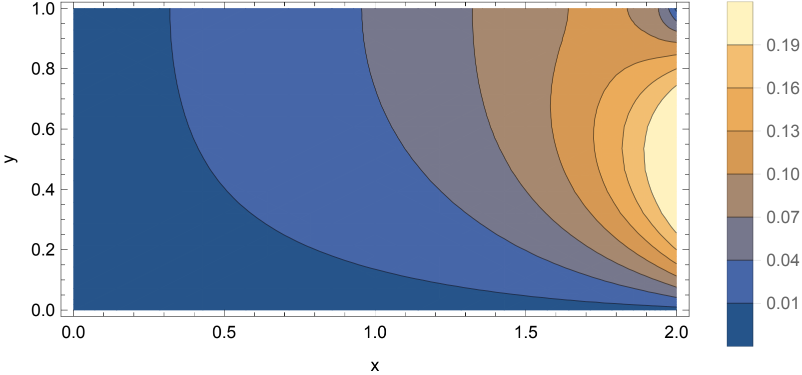
$$\begin{aligned} c_n \sinh((2n-1)\pi) &= 2\int_0^1 f(y) \sin \frac{(2n-1)\pi y}{2}\, dy \\ &= 2\int_0^1 y(1-y) \sin \frac{(2n-1)\pi y}{2}\, dy \\ &= \frac{8(4+(-1)^n(2n-1)\pi)}{(2n-1)^3\pi^3}. \end{aligned}$$

Solution (3 of 4)

The solution to the boundary value problem can be expressed as

$$u(x,y) = \frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{4 + (-1)^n (2n-1)\pi}{(2n-1)^3 \sinh((2n-1)\pi)} \sinh \frac{(2n-1)\pi x}{2} \sin \frac{(2n-1)\pi y}{2}.$$

Solution (4 of 4)



Example

Consider the boundary value problem:

$$\begin{aligned}\Delta u &= 0 \text{ for } 0 < x < 1 \text{ and } 0 < y < 1\\ u(x,0) &= u_y(x,1) = 0 \text{ for } 0 < x < 1\\ u(0,y) &= 0 \text{ for } 0 < y < 1\\ u_x(1,y) &= y \text{ for } 0 < y < 1.\end{aligned}$$

Find a solution to Laplace's equation with these mixed boundary conditions.

Solution (1 of 5)

Assuming a product solution $u(x, y) = X(x)Y(y)$, differentiating, and separating the variables implies the following ODEs and BCs (with σ constant).

$$X''(x) - \sigma X(x) = 0 \text{ with } X(0) = 0$$
$$Y''(y) + \sigma Y(y) = 0 \text{ with } Y(0) = 0 \text{ and } Y'(1) = 0.$$

The non-trivial solutions to the latter equation occur when

$$\sigma_n = \left(\frac{(2n-1)\pi}{2}\right)^2$$
$$Y_n(y) = \sin\left(\frac{(2n-1)\pi y}{2}\right)$$

for $n \in \mathbb{N}$.

Solution (2 of 5)

Solving the first BVP yields

$$X_n(x) = \sinh((2n-1)\pi x/2).$$

Hence the product solutions take the form

$$u_n(x,y) = \sinh\frac{(2n-1)\pi x}{2}\sin\frac{(2n-1)\pi y}{2} \text{ for } n \in \mathbb{N}.$$

Using the Principle of Superposition, the formal solution to Laplace's equation can be written as

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\frac{(2n-1)\pi x}{2}\sin\frac{(2n-1)\pi y}{2}.$$

The coefficients c_n must be determined from the nonhomogeneous BC.

Solution (3 of 5)

At the boundary where $x = 1$,

$$u_x(1, y) = \sum_{n=1}^{\infty} \frac{(2n-1)\pi}{2} c_n \cosh \frac{(2n-1)\pi}{2} \sin \frac{(2n-1)\pi y}{2} = y.$$

Therefore, c_n must be chosen so that

$$\begin{aligned}
\frac{(2n-1)\pi}{2} c_n \cosh \frac{(2n-1)\pi}{2} &= 2 \int_0^1 y \sin \frac{(2n-1)\pi y}{2} \, dy \\
&= \frac{8(-1)^{n+1}}{(2n-1)^2 \pi^2} \\
c_n &= \frac{16(-1)^{n+1}}{(2n-1)^3 \pi^3 \cosh \frac{(2n-1)\pi}{2}}
\end{aligned}$$

for $n \in \mathbb{N}$.

Solution (4 of 5)

The formal solution to the boundary value problem can be expressed as

$$u(x,y) = \frac{16}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3 \cosh \frac{(2n-1)\pi}{2}} \sinh \frac{(2n-1)\pi x}{2} \sin \frac{(2n-1)\pi y}{2}.$$

Solution (5 of 5)