

# Neumann Problems on Rectangles

MATH 467 *Partial Differential Equations*

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# Objectives

In this lesson we will learn:

- ▶ to solve Laplace's equation on two-dimensional domains with Neumann boundary conditions,
- ▶ to compare the solutions on domains with Dirichlet boundary conditions to solution on domains with Neumann boundary conditions.

# Boundary Value Problem

Consider Laplace's equation on the rectangle  
 $\Omega = \{(x, y) \mid 0 < x < a, 0 < y < b\}$  with Neumann boundary conditions:

$$\begin{aligned}\Delta u &= 0 \text{ for } (x, y) \in \Omega \\ u_y(x, 0) &= u_y(x, b) = 0 \text{ for } 0 < x < a \\ u_x(0, y) &= 0 \text{ for } 0 < y < b \\ u_x(a, y) &= f(y) \text{ for } 0 < y < b.\end{aligned}$$

**Physical Interpretation:** the steady-state heat distribution in  $\Omega$  when the rectangular region is insulated along its bottom, top, and left edges and there is a flow of heat on the right edge.

# Product Solutions

Assume  $u(x, y) = X(x)Y(y)$  solves Laplace's equation. Separation of variables induces the following ODEs for  $X(x)$  and  $Y(y)$ .

$$X''(x) - \sigma X(x) = 0 \text{ with } X'(0) = 0$$

$$Y''(y) + \sigma Y(y) = 0 \text{ with } Y'(0) = 0 = Y'(b),$$

where  $\sigma$  is a constant.

Taking the second BVP, the only nontrivial solutions are:

$$Y_n(y) = \cos \frac{n\pi y}{b}$$

$$\sigma_n = \frac{n^2\pi^2}{b^2}$$

for  $n = 0, 1, 2, \dots$

This implies  $X_n(x) = \cosh \frac{n\pi x}{b}$  for  $n = 0, 1, 2, \dots$

# Series Solution

The product solutions:

$$u_n(x, y) = \cosh \frac{n\pi x}{b} \cos \frac{n\pi y}{b} \text{ for } n = 0, 1, 2, \dots$$

solve Laplace's equation and satisfy the three homogeneous boundary conditions.

By the Principle of Superposition,

$$u(x, y) = b_0 + \sum_{n=1}^{\infty} b_n \cosh \frac{n\pi x}{b} \cos \frac{n\pi y}{b}.$$

The coefficients  $b_n$  can be determined from the remaining nonhomogeneous boundary condition.

# Determining the Coefficients

Differentiate the formal series solution and let  $x = a$ .

$$u_x(a, y) = \sum_{n=1}^{\infty} b_n \left( \frac{n\pi}{b} \right) \sinh \frac{n\pi a}{b} \cos \frac{n\pi y}{b} = f(y)$$

## Remarks:

- ▶ The coefficient  $b_0$  was lost during the differentiation.
- ▶ The infinite series can be regarded as a cosine series for  $f(y)$  if the integral of  $f$  over  $[0, b]$  vanishes, *i.e.*, if

$$\int_0^b f(y) dy = 0.$$

## Further Remarks

- ▶ If  $\int_0^b f(y) dy \neq 0$  then a solution to the BVP does not exist.
- ▶ Consider the physics:
  - ▶ If the definite integral vanishes then there is no net flux of heat across the boundary at  $x = a$  and hence a steady-state (time independent) heat distribution can evolve.
  - ▶ If the definite integral does not vanish, then there is a net flux of heat in or out of  $\Omega$  and no time independent temperature distribution can exist.
- ▶ Even if the definite integral vanishes, the solution can be determined only up to the addition of an arbitrary constant. Thus Laplace's equation on a rectangle with Neumann boundary conditions on all four edges has no unique solution.
- ▶ This type of boundary value problem is ill-posed.

Assuming  $\int_0^b f(y) dy = 0$

$$b_n = \frac{2}{n\pi \sinh \frac{n\pi a}{b}} \int_0^b f(y) \cos \frac{n\pi y}{b} dy,$$

for  $n \in \mathbb{N}$ .

$$u(x, y) = b_0 + \sum_{n=1}^{\infty} b_n \cosh \frac{n\pi x}{b} \cos \frac{n\pi y}{b}$$

where  $b_0$  is an arbitrary constant.



## Example

Find a solution to the Neumann boundary value problem on the unit square:

$$\Delta u = 0 \text{ for } 0 < x < 1 \text{ and } 0 < y < 1$$

$$u_y(x, 0) = u_y(x, 1) = 0 \text{ for } 0 < x < 1$$

$$u_x(0, y) = 0 \text{ for } 0 < y < 1$$

$$u_x(1, y) = y - 1/2 \text{ for } 0 < y < 1.$$

## Solution (1 of 2)

$$\text{Check: } \int_0^1 \left(y - \frac{1}{2}\right) dy = \left[\frac{y^2}{2} - \frac{y}{2}\right]_{y=0}^{y=1} = 0.$$

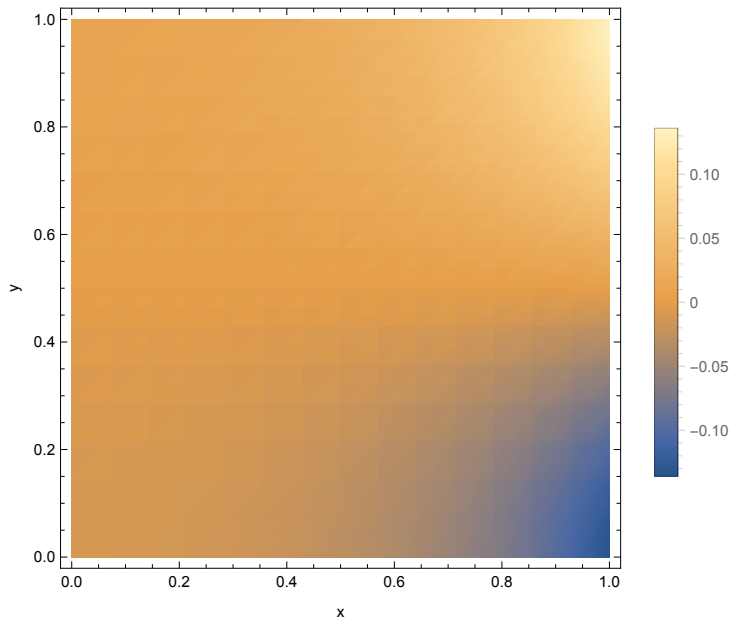
Using the Euler-Fourier coefficient formula:

$$\begin{aligned} b_n &= \frac{2}{n\pi \sinh(n\pi)} \int_0^1 \left(y - \frac{1}{2}\right) \cos(n\pi y) dy \\ &= \frac{-2}{n^2\pi^2 \sinh(n\pi)} \int_0^1 \sin(n\pi y) dy \\ &= \frac{2((-1)^n - 1)}{n^3\pi^3 \sinh(n\pi)}. \end{aligned}$$

$$u(x, y) = b_0 - \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{\cosh((2n-1)\pi x) \cos((2n-1)\pi y)}{(2n-1)^3 \sinh((2n-1)\pi)}$$

where  $b_0$  is an arbitrary constant.

## Solution (2 of 2)



# General Case

Consider Laplace's equation on a rectangle with Neumann BCs on all four edges.

$$u_{xx} + u_{yy} = 0 \text{ for } (x, y) \in R$$

$$u_x(0, y) = g_1(y) \text{ for } 0 < y < b$$

$$u_x(a, y) = g_2(y) \text{ for } 0 < y < b$$

$$u_y(x, 0) = f_1(x) \text{ for } 0 < x < a$$

$$u_y(x, b) = f_2(x) \text{ for } 0 < x < a$$

This BVP can be decomposed into four sub-problems with homogeneous boundary conditions on three edges.

# Sub-Problems

$$\begin{aligned}\Delta u_1 &= 0 \text{ for } (x, y) \in R \\ (u_1)_x(0, y) &= g_1(y) \text{ for } y \in (0, b) \\ (u_1)_x(a, y) &= 0 \text{ for } y \in (0, b) \\ (u_1)_y(x, 0) &= 0 \text{ for } x \in (0, a) \\ (u_1)_y(x, b) &= 0 \text{ for } x \in (0, a)\end{aligned}$$

$$\begin{aligned}\Delta u_2 &= 0 \text{ for } (x, y) \in R \\ (u_2)_x(0, y) &= 0 \text{ for } y \in (0, b) \\ (u_2)_x(a, y) &= g_2(y) \text{ for } y \in (0, b) \\ (u_2)_y(x, 0) &= 0 \text{ for } x \in (0, a) \\ (u_2)_y(x, b) &= 0 \text{ for } x \in (0, a)\end{aligned}$$

$$\begin{aligned}\Delta u_3 &= 0 \text{ for } (x, y) \in R \\ (u_3)_x(0, y) &= 0 \text{ for } y \in (0, b) \\ (u_3)_x(a, y) &= 0 \text{ for } y \in (0, b) \\ (u_3)_y(x, 0) &= f_1(y) \text{ for } x \in (0, a) \\ (u_3)_y(x, b) &= 0 \text{ for } x \in (0, a)\end{aligned}$$

$$\begin{aligned}\Delta u_4 &= 0 \text{ for } (x, y) \in R \\ (u_4)_x(0, y) &= 0 \text{ for } y \in (0, b) \\ (u_4)_x(a, y) &= 0 \text{ for } y \in (0, b) \\ (u_4)_y(x, 0) &= 0 \text{ for } x \in (0, a) \\ (u_4)_y(x, b) &= f_2(y) \text{ for } x \in (0, a)\end{aligned}$$

# Solutions to the Sub-Problems

$$u_1(x, y) = a_0 + \sum_{n=1}^{\infty} a_n \cosh \frac{n\pi(a-x)}{b} \cos \frac{n\pi y}{b}$$

$$u_2(x, y) = b_0 + \sum_{n=1}^{\infty} b_n \cosh \frac{n\pi x}{b} \cos \frac{n\pi y}{b}$$

$$u_3(x, y) = c_0 + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi x}{a} \cosh \frac{n\pi(b-y)}{a}$$

$$u_4(x, y) = d_0 + \sum_{n=1}^{\infty} d_n \cos \frac{n\pi x}{a} \cosh \frac{n\pi y}{a}$$

## Series Coefficients

Provided  $\int_0^b g_1(y) dy = \int_0^b g_2(y) dy = 0$  and  $\int_0^a f_1(x) dx = \int_0^a f_2(x) dx = 0$ , then

$$a_n = \frac{-2}{n\pi \sinh \frac{n\pi a}{b}} \int_0^b g_1(y) \cos \frac{n\pi y}{b} dy$$

$$b_n = \frac{2}{n\pi \sinh \frac{n\pi a}{b}} \int_0^b g_2(y) \cos \frac{n\pi y}{b} dy$$

$$c_n = \frac{-2}{n\pi \sinh \frac{n\pi b}{a}} \int_0^a f_1(x) \cos \frac{n\pi x}{a} dx$$

$$d_n = \frac{2}{n\pi \sinh \frac{n\pi b}{a}} \int_0^a f_2(x) \cos \frac{n\pi x}{a} dx.$$

The solution to the original BVP is then

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y).$$

# Neumann Problems on Disks

Consider Laplace's equation on the disk of radius  $a > 0$ :

$$\Delta u = 0 \text{ for } x^2 + y^2 < a^2$$

$$\frac{\partial u}{\partial \mathbf{n}}(x, y) = \phi(x, y) \text{ for } x^2 + y^2 = a^2.$$

$\partial u / \partial \mathbf{n}$  denotes the derivative in the direction of the unit outward normal vector to the boundary.

Convert to polar coordinates.

$$v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0 \text{ for } 0 < r < a \text{ and } -\infty < \theta < \infty$$

$$\frac{\partial v}{\partial r}(a, \theta) = \phi(a \cos \theta, a \sin \theta) = f(\theta) \text{ for } -\infty < \theta < \infty.$$



# Series Solution

The formal series solution can be written as

$$v(r, \theta) = d_0 + \sum_{n=1}^{\infty} r^n [c_n \cos(n\theta) + d_n \sin(n\theta)],$$

with coefficients  $d_0$ ,  $c_n$ , and  $d_n$  chosen such that

$$v_r(a, \theta) = \sum_{n=1}^{\infty} n a^{n-1} [c_n \cos(n\theta) + d_n \sin(n\theta)] = f(\theta).$$

A necessary condition for the solution to exist is

$$\int_{-\pi}^{\pi} f(\theta) d\theta = \int_{-\pi}^{\pi} \phi(a \cos \theta, a \sin \theta) d\theta = 0.$$

# Series Coefficients

$$c_n = \frac{a^{1-n}}{n\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta$$
$$d_n = \frac{a^{1-n}}{n\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta.$$

**Remark:** Coefficient  $d_0$  can be chosen arbitrarily and thus the solution to Laplace's equation on a disk with Neumann boundary conditions is not unique.

## Example

Find a bounded solution to Laplace's equation on  $\Omega = \{(r, \theta) \mid 0 \leq r < 1\}$  that satisfies the Neumann boundary condition,

$$u_r(1, \theta) = f(\theta) = \theta,$$

## Solution (1 of )

The solution can be written as

$$u(r, \theta) = d_0 + \sum_{n=1}^{\infty} r^n [c_n \cos(n\theta) + d_n \sin(n\theta)].$$

The boundary condition implies

$$u_r(1, \theta) = \sum_{n=1}^{\infty} n c_n \cos(n\theta) + n d_n \sin(n\theta) = \theta.$$

Applying the Euler-Fourier formula:

$$n c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \cos(n\theta) d\theta = 0$$

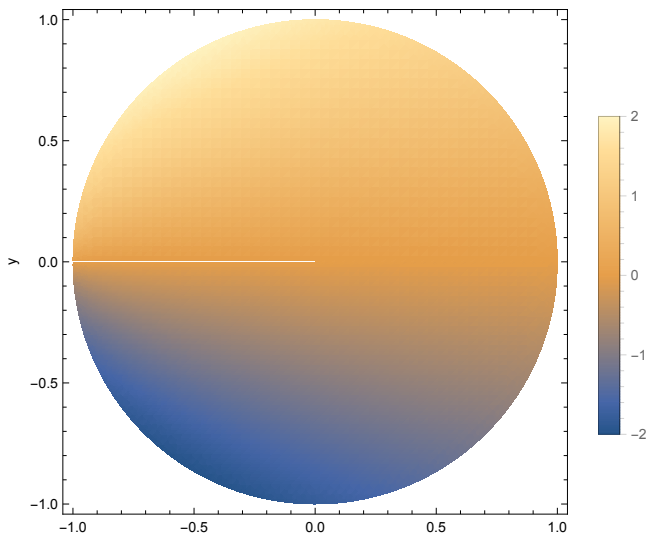
$$n d_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \sin(n\theta) d\theta$$

$$= \left[ \frac{-\theta}{n\pi} \cos(n\theta) \right]_{\theta=-\pi}^{\theta=\pi} + \frac{1}{n\pi} \int_{-\pi}^{\pi} \cos(n\theta) d\theta$$

$$d_n = \frac{-2(-1)^n}{n^2}.$$

## Solution (2 of )

$$u(r, \theta) = d_0 - 2 \sum_{n=1}^{\infty} \frac{(-1)^n r^n}{n^2} \sin(n\theta).$$



# Homework

- ▶ Read Sections 6.5 and 6.6
- ▶ Exercises: 20–23