Neumann Problems on Rectangles

Partial Differential Equations

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Objectives

In this lesson we will learn:

- to solve Laplace's equation on two-dimensional domains with Neumann boundary conditions,
- to compare the solutions on domains with Dirichlet boundary conditions to solution on domains with Neumann boundary conditions.

Boundary Value Problem

Consider Laplace's equation on the rectangle $\Omega = \{(x, y) \mid 0 < x < a,\ 0 < y < b\}$ with Neumann boundary conditions:

$$\begin{aligned}
\Delta u &= 0 \text{ for } (x, y) \in \Omega \\
u_y(x, 0) &= u_y(x, b) = 0 \text{ for } 0 < x < a \\
u_x(0, y) &= 0 \text{ for } 0 < y < b \\
u_x(a, y) &= f(y) \text{ for } 0 < y < b.
\end{aligned}$$

Physical Interpretation: the steady-state heat distribution in Ω when the rectangular region is insulated along its bottom, top, and left edges and there is a flow of heat on the right edge.

Product Solutions

Assume $u(x, y) = X(x)Y(y)$ solves Laplace's equation. Separation of variables induces the following ODEs for $X(x)$ and $Y(y)$.

$$X''(x) - \sigma X(x) = 0 \text{ with } X'(0) = 0$$
$$Y''(y) + \sigma Y(y) = 0 \text{ with } Y'(0) = 0 = Y'(b),$$

where σ is a constant.

Taking the second BVP, the only nontrivial solutions are:

$$Y_n(y) = \cos \frac{n\pi y}{b}$$
$$\sigma_n = \frac{n^2\pi^2}{b^2}$$

for $n = 0, 1, 2, \ldots$.

This implies $X_n(x) = \cosh \dfrac{n\pi x}{b}$ for $n = 0, 1, 2, \ldots$.

Series Solution

The product solutions:

$$u_n(x,y) = \cosh \frac{n\pi x}{b} \cos \frac{n\pi y}{b} \text{ for } n = 0, 1, 2, \ldots$$

solve Laplace's equation and satisfy the three homogeneous boundary conditions.

By the Principle of Superposition,

$$u(x,y) = b_0 + \sum_{n=1}^{\infty} b_n \cosh \frac{n\pi x}{b} \cos \frac{n\pi y}{b}.$$

The coefficients b_n can be determined from the remaining nonhomogeneous boundary condition.

Determining the Coefficients

Differentiate the formal series solution and let $x = a$.

$$u_x(a, y) = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi}{b}\right) \sinh \frac{n\pi a}{b} \cos \frac{n\pi y}{b} = f(y)$$

Remarks:

- The coefficient b_0 was lost during the differentiation.
- The infinite series can be regarded as a cosine series for $f(y)$ if the integral of f over $[0, b]$ vanishes, *i.e.*, if

$$\int_0^b f(y)\, dy = 0.$$

Further Remarks

- If $\int_0^b f(y)\,dy \neq 0$ then a solution to the BVP does not exist.
- Consider the physics:
  - If the definite integral vanishes then there is no net flux of heat across the boundary at $x = a$ and hence a steady-state (time independent) heat distribution can evolve.
  - If the definite integral does not vanish, then there is a net flux of heat in or out of Ω and no time independent temperature distribution can exist.
- Even if the definite integral vanishes, the solution can be determined only up to the addition of an arbitrary constant. Thus Laplace's equation on a rectangle with Neumann boundary conditions on all four edges has no unique solution.
- This type of boundary value problem is ill-posed.

Assuming $\int_0^b f(y)\,dy = 0$

$$f(y) = \sum_{n=1}^{\infty} b_m \left(\frac{m\pi}{b}\right) \sinh\frac{m\pi a}{b} \cos\frac{m\pi y}{b}$$

Multiply both sides by $\cos\dfrac{n\pi y}{b}$ and integrate for $0 \le y \le b$.

$$\begin{aligned}
\int_0^b f(y)\cos\frac{n\pi y}{b}\,dy &= \sum_{n=1}^{\infty} b_m \left(\frac{m\pi}{b}\right) \sinh\frac{m\pi a}{b} \int_0^b \cos\frac{m\pi y}{b}\cos\frac{n\pi y}{b}\,dy \\
&= b_n \left(\frac{n\pi}{b}\right) \sinh\frac{n\pi a}{b} \left(\frac{b}{2}\right) \quad \text{(orthogonality)} \\
b_n &= \frac{2}{n\pi \sinh\frac{n\pi a}{b}} \int_0^b f(y)\cos\frac{n\pi y}{b}\,dy,
\end{aligned}$$

for $n \in \mathbb{N}$.

$$u(x,y) = b_0 + \sum_{n=1}^{\infty} b_n \cosh\frac{n\pi x}{b} \cos\frac{n\pi y}{b}$$

where b_0 is an arbitrary constant.

Example

Find a solution to the Neumann boundary value problem on the unit square:

$$\begin{aligned}
\Delta u &= 0 \text{ for } 0 < x < 1 \text{ and } 0 < y < 1 \\
u_y(x,0) &= u_y(x,1) = 0 \text{ for } 0 < x < 1 \\
u_x(0,y) &= 0 \text{ for } 0 < y < 1 \\
u_x(1,y) &= y - 1/2 \text{ for } 0 < y < 1.
\end{aligned}$$

Solution (1 of 2)

Check: $\int_0^1 \left(y - \frac{1}{2}\right)\, dy = \left[\frac{y^2}{2} - \frac{y}{2}\right]_{y=0}^{y=1} = 0.$

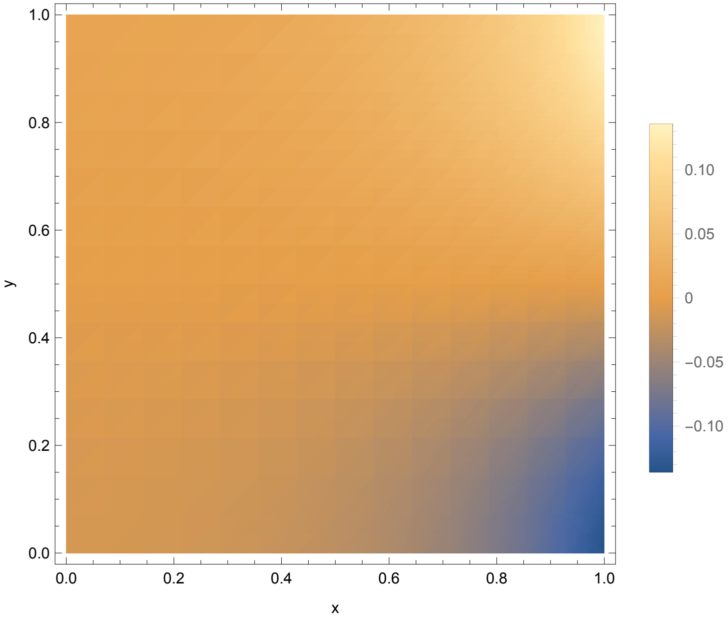
Using the Euler-Fourier coefficient formula:

$$\begin{aligned} b_n &= \frac{2}{n\pi \sinh(n\pi)} \int_0^1 \left(y - \frac{1}{2}\right) \cos(n\pi y)\, dy \\ &= \frac{-2}{n^2\pi^2 \sinh(n\pi)} \int_0^1 \sin(n\pi y)\, dy \\ &= \frac{2((-1)^n - 1)}{n^3\pi^3 \sinh(n\pi)}. \end{aligned}$$

$$u(x,y) = b_0 - \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{\cosh((2n-1)\pi x)\cos((2n-1)\pi y)}{(2n-1)^3 \sinh((2n-1)\pi)}$$

where b_0 is an arbitrary constant.

Solution (2 of 2)



General Case

Consider Laplace's equation on a rectangle with Neumann BCs on all four edges.

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \text{ for } (x, y) \in R \\ u_x(0, y) &= g_1(y) \text{ for } 0 < y < b \\ u_x(a, y) &= g_2(y) \text{ for } 0 < y < b \\ u_y(x, 0) &= f_1(x) \text{ for } 0 < x < a \\ u_y(x, b) &= f_2(x) \text{ for } 0 < x < a \end{aligned}$$

This BVP can be decomposed into four sub-problems with homogeneous boundary conditions on three edges.

Sub-Problems

$$
\begin{aligned}
\Delta u_1 &= 0 \text{ for } (x,y) \in R \\
(u_1)_x(0,y) &= g_1(y) \text{ for } y \in (0,b) \\
(u_1)_x(a,y) &= 0 \text{ for } y \in (0,b) \\
(u_1)_y(x,0) &= 0 \text{ for } x \in (0,a) \\
(u_1)_y(x,b) &= 0 \text{ for } x \in (0,a)
\end{aligned}
$$

$$
\begin{aligned}
\Delta u_2 &= 0 \text{ for } (x,y) \in R \\
(u_2)_x(0,y) &= 0 \text{ for } y \in (0,b) \\
(u_2)_x(a,y) &= g_2(y) \text{ for } y \in (0,b) \\
(u_2)_y(x,0) &= 0 \text{ for } x \in (0,a) \\
(u_2)_y(x,b) &= 0 \text{ for } x \in (0,a)
\end{aligned}
$$

$$
\begin{aligned}
\Delta u_3 &= 0 \text{ for } (x,y) \in R \\
(u_3)_x(0,y) &= 0 \text{ for } y \in (0,b) \\
(u_3)_x(a,y) &= 0 \text{ for } y \in (0,b) \\
(u_3)_y(x,0) &= f_1(y) \text{ for } x \in (0,a) \\
(u_3)_y(x,b) &= 0 \text{ for } x \in (0,a)
\end{aligned}
$$

$$
\begin{aligned}
\Delta u_4 &= 0 \text{ for } (x,y) \in R \\
(u_4)_x(0,y) &= 0 \text{ for } y \in (0,b) \\
(u_4)_x(a,y) &= 0 \text{ for } y \in (0,b) \\
(u_4)_y(x,0) &= 0 \text{ for } x \in (0,a) \\
(u_4)_y(x,b) &= f_2(y) \text{ for } x \in (0,a)
\end{aligned}
$$

Solutions to the Sub-Problems

$$u_1(x,y) = a_0 + \sum_{n=1}^{\infty} a_n \cosh\frac{n\pi(a-x)}{b}\cos\frac{n\pi y}{b}$$

$$u_2(x,y) = b_0 + \sum_{n=1}^{\infty} b_n \cosh\frac{n\pi x}{b}\cos\frac{n\pi y}{b}$$

$$u_3(x,y) = c_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi x}{a}\cosh\frac{n\pi(b-y)}{a}$$

$$u_4(x,y) = d_0 + \sum_{n=1}^{\infty} d_n \cos\frac{n\pi x}{a}\cosh\frac{n\pi y}{a}$$

Series Coefficients

Provided $\int_0^b g_1(y)\,dy = \int_0^b g_2(y)\,dy = 0$ and $\int_0^a f_1(x)\,dx = \int_0^a f_2(x)\,dx = 0$, then

$$
\begin{aligned}
a_n &= \frac{-2}{n\pi \sinh \frac{n\pi a}{b}} \int_0^b g_1(y) \cos \frac{n\pi y}{b}\,dy \\
b_n &= \frac{2}{n\pi \sinh \frac{n\pi a}{b}} \int_0^b g_2(y) \cos \frac{n\pi y}{b}\,dy \\
c_n &= \frac{-2}{n\pi \sinh \frac{n\pi b}{a}} \int_0^a f_1(x) \cos \frac{n\pi x}{a}\,dx \\
d_n &= \frac{2}{n\pi \sinh \frac{n\pi b}{a}} \int_0^a f_2(x) \cos \frac{n\pi x}{a}\,dx.
\end{aligned}
$$

The solution to the original BVP is then

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y).$$

Neumann Problems on Disks

Consider Laplace's equation on the disk of radius $a > 0$:

$$\Delta u = 0 \text{ for } x^2 + y^2 < a^2$$
$$\frac{\partial u}{\partial \mathbf{n}}(x, y) = \phi(x, y) \text{ for } x^2 + y^2 = a^2.$$

$\partial u / \partial \mathbf{n}$ denotes the derivative in the direction of the unit outward normal vector to the boundary.

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Convert to polar coordinates.

$$v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0 \text{ for } 0 < r < a \text{ and } -\infty < \theta < \infty$$
$$\frac{\partial v}{\partial r}(a, \theta) = \phi(a \cos\theta, a \sin\theta) = f(\theta) \text{ for } -\infty < \theta < \infty.$$

Series Solution

The formal series solution can be written as

$$v(r, \theta) = d_0 + \sum_{n=1}^{\infty} r^n [c_n \cos(n\theta) + d_n \sin(n\theta)],$$

with coefficients d_0, c_n, and d_n chosen such that

$$v_r(a, \theta) = \sum_{n=1}^{\infty} n a^{n-1} [c_n \cos(n\theta) + d_n \sin(n\theta)] = f(\theta).$$

A necessary condition for the solution to exist is

$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = \int_{-\pi}^{\pi} \phi(a \cos\theta, a \sin\theta)\, d\theta = 0.$$

Series Coefficients (1 of 2)

$$f(\theta) = \sum_{m=1}^{\infty} m a^{m-1} [c_m \cos(m\theta) + d_m \sin(m\theta)]$$

Multiply both sides by $\cos(n\theta)$ and integrate over $[-\pi, \pi]$.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta &= \sum_{m=1}^{\infty} m a^{m-1} \left[c_m \int_{-\pi}^{\pi} \cos(m\theta) \cos(n\theta)\, d\theta \right. \\
&\qquad \left. + \, d_m \int_{-\pi}^{\pi} \sin(m\theta) \cos(n\theta)\, d\theta \right] \\
&= n a^{n-1} c_n \pi \text{ (orthogonality)} \\
c_n &= \frac{a^{1-n}}{n\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta
\end{aligned}$$

for $n \in \mathbb{N}$.

Series Coefficients (2 of 2)

$$f(\theta) = \sum_{m=1}^{\infty} m a^{m-1} [c_m \cos(m\theta) + d_m \sin(m\theta)]$$

Multiply both sides by $\sin(n\theta)$ and integrate over $[-\pi, \pi]$.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta &= \sum_{m=1}^{\infty} m a^{m-1} \left[c_m \int_{-\pi}^{\pi} \cos(m\theta) \sin(n\theta)\, d\theta \right. \\
&\qquad \left. + \, d_m \int_{-\pi}^{\pi} \sin(m\theta) \sin(n\theta)\, d\theta \right] \\
&= n a^{n-1} d_n \pi \text{ (orthogonality)} \\
d_n &= \frac{a^{1-n}}{n\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta
\end{aligned}$$

for $n \in \mathbb{N}$.

Remark: Coefficient d_0 can be chosen arbitrarily and thus the solution to Laplace's equation on a disk with Neumann boundary conditions is not unique.

Example

Find a bounded solution to Laplace's equation on $\Omega = \{(r, \theta) \,|\, 0 \le r < 1\}$ that satisfies the Neumann boundary condition,

$$u_r(1, \theta) = f(\theta) = \theta,$$

Solution (1 of 2)

The solution can be written as

$$u(r,\theta) = d_0 + \sum_{n=1}^{\infty} r^n [c_n \cos(n\theta) + d_n \sin(n\theta)].$$

The boundary condition implies

$$u_r(1,\theta) = \sum_{n=1}^{\infty} n\, c_n \cos(n\theta) + n\, d_n \sin(n\theta)] = \theta.$$

Applying the Euler-Fourier formula:

$$\begin{aligned} n\, c_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \theta\, \cos(n\theta)\, d\theta = 0 \\ n\, d_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \theta\, \sin(n\theta)\, d\theta \\ &= \left[\frac{-\theta}{n\pi}\cos(n\theta)\right]_{\theta=-\pi}^{\theta=\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi} \cos(n\theta)\, d\theta \\ d_n &= \frac{-2(-1)^n}{n^2}. \end{aligned}$$

Solution (2 of 2)

$$u(r, \theta) = d_0 - 2 \sum_{n=1}^{\infty} \frac{(-1)^n r^n}{n^2} \sin(n\theta).$$