

# Poisson's Formula

MATH 467 *Partial Differential Equations*

J Robert Buchanan

Department of Mathematics

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# Objectives

In this lesson we will learn about

- ▶ a closed form solution to Laplace's equation on a disk,
- ▶ properties of harmonic functions.

## Background

Consider Laplace's equation (in polar coordinates) on a disk:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{for } 0 < r < a \text{ and } -\infty < \theta < \infty$$
$$u(a, \theta) = f(\theta)$$

The solution can be expressed as

$$u(r, \theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n [\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)]$$

where

$$\alpha_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta$$

$$\beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta.$$

## Closed Form Solution (1 of 3)

We can express the Fourier series solution as

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[ \frac{\cos(n\theta)}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt \right] \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[ \frac{\sin(n\theta)}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt \right] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) [\cos(n(\theta + t)) + \cos(n(\theta - t))] dt \right] \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) [\cos(n(\theta - t)) - \cos(n(\theta + t))] dt \right] \end{aligned}$$

## Closed Form Solution (2 of 3)

$$\begin{aligned}u(r, \theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(n(\theta - t)) dt \\&= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} f(t) dt + \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n f(t) \cos(n(\theta - t)) dt \\&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n(\theta - t)) \right] dt \\&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \operatorname{Re} \left( e^{in(\theta - t)} \right) \right] dt \\&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[ \frac{1}{2} + \operatorname{Re} \left( \sum_{n=1}^{\infty} \left( \frac{re^{i(\theta - t)}}{a} \right)^n \right) \right] dt\end{aligned}$$

## Closed Form Solution (3 of 3)

$$\begin{aligned}u(r, \theta) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[ -\frac{1}{2} + \operatorname{Re} \left( \sum_{n=0}^{\infty} \left( \frac{re^{i(\theta-t)}}{a} \right)^n \right) \right] dt \\&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[ -\frac{1}{2} + \operatorname{Re} \left( \frac{1}{1 - \frac{re^{i(\theta-t)}}{a}} \right) \right] dt \\&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[ -\frac{1}{2} + \operatorname{Re} \left( \frac{1}{1 - \frac{r}{a} \cos(\theta - t) - i \frac{r}{a} \sin(\theta - t)} \right) \right] dt \\&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[ -\frac{1}{2} + \frac{1 - \frac{r}{a} \cos(\theta - t)}{1 + \frac{r^2}{a^2} - \frac{2r}{a} \cos(\theta - t)} \right] dt \\u(r, \theta) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(t) \left( \frac{1}{2} - \frac{r^2}{2a^2} \right)}{1 + \frac{r^2}{a^2} - \frac{2r}{a} \cos(\theta - t)} dt\end{aligned}$$

This is known as **Poisson's integral formula**.

# Main Result

## Theorem

*If  $u(r, \theta)$  is a solution to Laplace's equation in polar coordinates on the disk:*

$$\Delta u = 0 \quad \text{for } 0 < r < a \text{ and } -\infty < \theta < \infty$$

$$u(a, \theta) = f(\theta)$$

*then*

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)(a^2 - r^2)}{a^2 - 2ar \cos(t - \theta) + r^2} dt.$$

# Mean Value at the Origin

## Corollary

*If  $u(r, \theta)$  is a solution to Laplace's equation in polar coordinates on the disk:*

$$\begin{aligned}\Delta u &= 0 \quad \text{for } 0 < r < a \text{ and } -\infty < \theta < \infty \\ u(a, \theta) &= f(\theta)\end{aligned}$$

*then*

$$u(0, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt.$$



# Proof

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)(a^2 - r^2)}{a^2 - 2ar \cos(t - \theta) + r^2} dt$$

$$u(0, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

# Mean Value Result

## Theorem

*Let  $u(x, y)$  be a harmonic function on  $\Omega$ , an open, connected subset of  $\mathbb{R}^2$ , let  $(x_0, y_0) \in \Omega$ , and let  $a > 0$  be chosen so that*

$$D_a(x_0, y_0) = \{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 < a^2\} \subset \Omega$$

*and*

$$C_a(x_0, y_0) = \{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 = a^2\} \subset \Omega,$$

*then*

$$u(x_0, y_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + a \cos \theta, y_0 + a \sin \theta) d\theta.$$

*That is  $u(x_0, y_0)$  is the mean value of  $u$  on  $C_a(x_0, y_0)$ , the boundary of  $D_a(x_0, y_0)$ .*

## Proof (1 of 2)

- ▶ Let  $\xi = x - x_0$  and  $\eta = y - y_0$ , then  
 $D_a(x = x_0, y = y_0) = D_a(\xi = 0, \eta = 0)$ .
- ▶ Define the function  $U(\xi, \eta) = u(x_0 + \xi, y_0 + \eta)$ .
- ▶ Since  $u$  is harmonic on  $\Omega$  then by the chain rule

$$U_{\xi\xi} + U_{\eta\eta} = 0.$$

## Proof (2 of 2)

- Using Poisson's formula, then for  $(\xi, \eta) \in D_a(0, 0)$

$$\begin{aligned}U(\xi, \eta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(a^2 - r^2)U(a \cos t, a \sin t)}{a^2 - 2ar \cos(t - \theta) + r^2} dt \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(a^2 - r^2)u(x_0 + a \cos t, y_0 + a \sin t)}{a^2 - 2ar \cos(t - \theta) + r^2} dt\end{aligned}$$

- If  $(\xi, \eta) = (0, 0)$

$$U(0, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + a \cos t, y_0 + a \sin t) dt$$

which is equivalent to

$$u(x_0, y_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + a \cos t, y_0 + a \sin t) dt.$$

# Maximum/Minimum Principle

## Theorem

Let  $\Omega$  be an open, connected, and bounded domain in  $\mathbb{R}^2$  with boundary  $\partial\Omega$ . Assume that  $u(x, y) \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$  where  $\overline{\Omega} = \Omega \cup \partial\Omega$ , then

- ▶ if  $u_{xx} + u_{yy} \geq 0$  for all  $(x, y) \in \Omega$ , then

$$\max_{(x,y) \in \overline{\Omega}} u(x, y) = \max_{(x,y) \in \partial\Omega} u(x, y),$$

- ▶ if  $u_{xx} + u_{yy} \leq 0$  for all  $(x, y) \in \Omega$ , then

$$\min_{(x,y) \in \overline{\Omega}} u(x, y) = \min_{(x,y) \in \partial\Omega} u(x, y).$$

# Maximum/Minimum Principle for Laplace's Equation

## Corollary

*Let  $\Omega$  be an open, connected, and bounded domain in  $\mathbb{R}^2$ . Assume that  $u(x, y) \in C^2(\Omega) \cap C(\overline{\Omega})$  where  $\overline{\Omega} = \Omega \cup \partial\Omega$  and  $u(x, y)$  satisfies Laplace's equation on  $\Omega$ , then*

$$\max_{(x,y) \in \overline{\Omega}} u(x, y) = \max_{(x,y) \in \partial\Omega} u(x, y)$$

*and*

$$\min_{(x,y) \in \overline{\Omega}} u(x, y) = \min_{(x,y) \in \partial\Omega} u(x, y).$$

# Uniqueness of Solutions

## Theorem (Uniqueness)

*Suppose that  $\Omega$  is an open, connected, and bounded domain in  $\mathbb{R}^2$  with  $f(x, y) \in \mathcal{C}(\Omega)$  and  $\phi(x, y) \in \mathcal{C}(\partial\Omega)$ . If  $u(x, y) \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$  where  $\overline{\Omega} = \Omega \cup \partial\Omega$  is a solution to the boundary value problem,*

$$\begin{aligned}\Delta u &= f(x, y) \text{ for } (x, y) \in \Omega \\ u(x, y) &= \phi(x, y) \text{ for } (x, y) \in \partial\Omega,\end{aligned}$$

*then this solution is unique.*

# Homework

- ▶ Read Section 6.8–6.9
- ▶ Exercises: 28