

Poisson's Formula

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will learn about

- ▶ a closed form solution to Laplace's equation on a disk,
- ▶ properties of harmonic functions.

Background

Consider Laplace's equation (in polar coordinates) on a disk:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{for } 0 < r < a \text{ and } -\infty < \theta < \infty$$

$$u(a, \theta) = f(\theta)$$

The solution can be expressed as

$$u(r, \theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n [\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)]$$

where

$$\alpha_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta$$

$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) \, d\theta$$

$$\beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) \, d\theta.$$

Closed Form Solution (1 of 3)

We can express the Fourier series solution as

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, dt + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[\frac{\cos(n\theta)}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \, dt \right] \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[\frac{\sin(n\theta)}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \, dt \right] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, dt \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) [\cos(n(\theta + t)) + \cos(n(\theta - t))] \, dt \right] \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) [\cos(n(\theta - t)) - \cos(n(\theta + t))] \, dt \right] \end{aligned}$$

Closed Form Solution (2 of 3)

$$\begin{aligned}
 u(r, \theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(n(\theta - t)) dt \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} f(t) dt + \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n f(t) \cos(n(\theta - t)) dt \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n(\theta - t)) \right] dt \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \operatorname{Re} \left(e^{in(\theta - t)} \right) \right] dt \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \operatorname{Re} \left(\sum_{n=1}^{\infty} \left(\frac{re^{i(\theta - t)}}{a} \right)^n \right) \right] dt
 \end{aligned}$$

Closed Form Solution (3 of 3)

$$\begin{aligned}
 u(r, \theta) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[-\frac{1}{2} + \operatorname{Re} \left(\sum_{n=0}^{\infty} \left(\frac{re^{i(\theta - t)}}{a} \right)^n \right) \right] dt \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[-\frac{1}{2} + \operatorname{Re} \left(\frac{1}{1 - \frac{re^{i(\theta - t)}}{a}} \right) \right] dt \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[-\frac{1}{2} + \operatorname{Re} \left(\frac{1}{1 - \frac{r}{a} \cos(\theta - t) - i \frac{r}{a} \sin(\theta - t)} \right) \right] dt \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[-\frac{1}{2} + \frac{1 - \frac{r}{a} \cos(\theta - t)}{1 + \frac{r^2}{a^2} - \frac{2r}{a} \cos(\theta - t)} \right] dt \\
 u(r, \theta) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(t) \left(\frac{1}{2} - \frac{r^2}{2a^2} \right)}{1 + \frac{r^2}{a^2} - \frac{2r}{a} \cos(\theta - t)} dt
 \end{aligned}$$

This is known as **Poisson's integral formula**.

Main Result

Mean Value at the Origin

Theorem

If $u(r, \theta)$ is a solution to Laplace's equation in polar coordinates on the disk:

$$\begin{aligned}
 \Delta u &= 0 \quad \text{for } 0 < r < a \text{ and } -\infty < \theta < \infty \\
 u(a, \theta) &= f(\theta)
 \end{aligned}$$

then

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)(a^2 - r^2)}{a^2 - 2ar \cos(t - \theta) + r^2} dt.$$

Corollary

If $u(r, \theta)$ is a solution to Laplace's equation in polar coordinates on the disk:

$$\begin{aligned}
 \Delta u &= 0 \quad \text{for } 0 < r < a \text{ and } -\infty < \theta < \infty \\
 u(a, \theta) &= f(\theta)
 \end{aligned}$$

then

$$u(0, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt.$$

Proof

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)(a^2 - r^2)}{a^2 - 2ar \cos(t - \theta) + r^2} dt$$

$$u(0, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

Proof (1 of 2)

- ▶ Let $\xi = x - x_0$ and $\eta = y - y_0$, then $D_a(x = x_0, y = y_0) = D_a(\xi = 0, \eta = 0)$.
- ▶ Define the function $U(\xi, \eta) = u(x_0 + \xi, y_0 + \eta)$.
- ▶ Since u is harmonic on Ω then by the chain rule

$$U_{\xi\xi} + U_{\eta\eta} = 0.$$

Mean Value Result

Theorem

Let $u(x, y)$ be a harmonic function on Ω , an open, connected subset of \mathbb{R}^2 , let $(x_0, y_0) \in \Omega$, and let $a > 0$ be chosen so that

$$D_a(x_0, y_0) = \{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 < a^2\} \subset \Omega$$

and

$$C_a(x_0, y_0) = \{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 = a^2\} \subset \Omega,$$

then

$$u(x_0, y_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + a \cos \theta, y_0 + a \sin \theta) d\theta.$$

That is $u(x_0, y_0)$ is the mean value of u on $C_a(x_0, y_0)$, the boundary of $D_a(x_0, y_0)$.

Proof (2 of 2)

- ▶ Using Poisson's formula, then for $(\xi, \eta) \in D_a(0, 0)$

$$U(\xi, \eta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(a^2 - r^2)U(a \cos t, a \sin t)}{a^2 - 2ar \cos(t - \theta) + r^2} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(a^2 - r^2)u(x_0 + a \cos t, y_0 + a \sin t)}{a^2 - 2ar \cos(t - \theta) + r^2} dt$$

- ▶ If $(\xi, \eta) = (0, 0)$

$$U(0, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + a \cos t, y_0 + a \sin t) dt$$

which is equivalent to

$$u(x_0, y_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + a \cos t, y_0 + a \sin t) dt.$$

Maximum/Minimum Principle

Maximum/Minimum Principle for Laplace's Equation

Theorem

Let Ω be an open, connected, and bounded domain in \mathbb{R}^2 with boundary $\partial\Omega$. Assume that $u(x, y) \in C^2(\Omega) \cap C(\bar{\Omega})$ where $\bar{\Omega} = \Omega \cup \partial\Omega$, then

► if $u_{xx} + u_{yy} \geq 0$ for all $(x, y) \in \Omega$, then

$$\max_{(x,y) \in \bar{\Omega}} u(x, y) = \max_{(x,y) \in \partial\Omega} u(x, y),$$

► if $u_{xx} + u_{yy} \leq 0$ for all $(x, y) \in \Omega$, then

$$\min_{(x,y) \in \bar{\Omega}} u(x, y) = \min_{(x,y) \in \partial\Omega} u(x, y).$$

Corollary

Let Ω be an open, connected, and bounded domain in \mathbb{R}^2 . Assume that $u(x, y) \in C^2(\Omega) \cap C(\bar{\Omega})$ where $\bar{\Omega} = \Omega \cup \partial\Omega$ and $u(x, y)$ satisfies Laplace's equation on Ω , then

$$\max_{(x,y) \in \bar{\Omega}} u(x, y) = \max_{(x,y) \in \partial\Omega} u(x, y)$$

and

$$\min_{(x,y) \in \bar{\Omega}} u(x, y) = \min_{(x,y) \in \partial\Omega} u(x, y).$$

Uniqueness of Solutions

Homework

Theorem (Uniqueness)

Suppose that Ω is an open, connected, and bounded domain in \mathbb{R}^2 with $f(x, y) \in C(\Omega)$ and $\phi(x, y) \in C(\partial\Omega)$. If $u(x, y) \in C^2(\Omega) \cap C(\bar{\Omega})$ where $\bar{\Omega} = \Omega \cup \partial\Omega$ is a solution to the boundary value problem,

$$\begin{aligned} \Delta u &= f(x, y) \text{ for } (x, y) \in \Omega \\ u(x, y) &= \phi(x, y) \text{ for } (x, y) \in \partial\Omega, \end{aligned}$$

then this solution is unique.

- Read Section 6.8–6.9
- Exercises: 28