Poisson's Formula

Partial Differential Equations

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Objectives

In this lesson we will learn about

- a closed form solution to Laplace's equation on a disk,
- properties of harmonic functions.

Background

Consider Laplace's equation (in polar coordinates) on a disk:

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \text{ for } 0 < r < a \text{ and } -\infty < \theta < \infty$$
$$u(a, \theta) = f(\theta)$$

The solution can be expressed as

$$u(r, \theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n [\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)]$$

where

$$\alpha_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$
$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta$$
$$\beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta.$$

Closed Form Solution (1 of 3)

We can express the Fourier series solution as

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[\frac{\cos(n\theta)}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt\right] \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[\frac{\sin(n\theta)}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)[\cos(n(\theta+t)) + \cos(n(\theta-t))]\,dt\right] \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)[\cos(n(\theta-t)) - \cos(n(\theta+t))]\,dt\right]
\end{aligned}$$

Closed Form Solution (2 of 3)

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n(\theta-t))\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2} f(t)\,dt + \frac{1}{\pi}\int_{-\pi}^{\pi} \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n f(t)\cos(n(\theta-t))\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-t))\right] dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \operatorname{Re}\left(e^{in(\theta-t)}\right)\right] dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \operatorname{Re}\left(\sum_{n=1}^{\infty}\left(\frac{re^{i(\theta-t)}}{a}\right)^n\right)\right] dt
\end{aligned}
$$

Closed Form Solution (3 of 3)

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[-\frac{1}{2} + \operatorname{Re}\left(\sum_{n=0}^{\infty}\left(\frac{re^{i(\theta-t)}}{a}\right)^{n}\right)\right] dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[-\frac{1}{2} + \operatorname{Re}\left(\frac{1}{1-\frac{re^{i(\theta-t)}}{a}}\right)\right] dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[-\frac{1}{2} + \operatorname{Re}\left(\frac{1}{1-\frac{r}{a}\cos(\theta-t) - i\frac{r}{a}\sin(\theta-t)}\right)\right] dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[-\frac{1}{2} + \frac{1-\frac{r}{a}\cos(\theta-t)}{1+\frac{r^2}{a^2}-\frac{2r}{a}\cos(\theta-t)}\right] dt \\
u(r,\theta) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t)\left(\frac{1}{2}-\frac{r^2}{2a^2}\right)}{1+\frac{r^2}{a^2}-\frac{2r}{a}\cos(\theta-t)}\, dt
\end{aligned}
$$

This is known as **Poisson's integral formula**.

Main Result

Theorem

If $u(r, \theta)$ is a solution to Laplace's equation in polar coordinates on the disk:

$$\begin{aligned} \Delta u &= 0 \text{ for } 0 < r < a \text{ and } -\infty < \theta < \infty \\ u(a, \theta) &= f(\theta) \end{aligned}$$

then

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)(a^2 - r^2)}{a^2 - 2ar\cos(t - \theta) + r^2}\, dt.$$

Mean Value at the Origin

Corollary

If $u(r,\theta)$ is a solution to Laplace's equation in polar coordinates on the disk:

$$\begin{aligned}\Delta u &= 0 \text{ for } 0 < r < a \text{ and } -\infty < \theta < \infty \\ u(a,\theta) &= f(\theta)\end{aligned}$$

then

$$u(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt.$$

Proof

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(t)(a^2 - r^2)}{a^2 - 2ar\cos(t-\theta) + r^2}\,dt$$

$$u(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

Mean Value Result

Theorem

Let $u(x, y)$ be a harmonic function on Ω, an open, connected subset of $\mathbb{R}^2$, let $(x_0, y_0) \in \Omega$, and let $a > 0$ be chosen so that

$$D_a(x_0, y_0) = \{(x, y) \,|\, (x - x_0)^2 + (y - y_0)^2 < a^2\} \subset \Omega$$

and

$$C_a(x_0, y_0) = \{(x, y) \,|\, (x - x_0)^2 + (y - y_0)^2 = a^2\} \subset \Omega,$$

then

$$u(x_0, y_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + a\cos\theta, y_0 + a\sin\theta)\, d\theta.$$

That is $u(x_0, y_0)$ is the mean value of u on $C_a(x_0, y_0)$, the boundary of $D_a(x_0, y_0)$.

Proof (1 of 2)

- Let $\xi = x - x_0$ and $\eta = y - y_0$, then $D_a(x = x_0, y = y_0) = D_a(\xi = 0, \eta = 0)$.
- Define the function $U(\xi, \eta) = u(x_0 + \xi, y_0 + \eta)$.
- Since u is harmonic on Ω then by the chain rule

$$U_{\xi\xi} + U_{\eta\eta} = 0.$$

Proof (2 of 2)

- Using Poisson's formula, then for $(\xi, \eta) \in D_a(0,0)$

$$U(\xi,\eta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(a^2 - r^2)U(a\cos t, a\sin t)}{a^2 - 2ar\cos(t-\theta) + r^2}\,dt$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(a^2 - r^2)u(x_0 + a\cos t, y_0 + a\sin t)}{a^2 - 2ar\cos(t-\theta) + r^2}\,dt$$

- If $(\xi, \eta) = (0,0)$

$$U(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x_0 + a\cos t, y_0 + a\sin t)\,dt$$

which is equivalent to

$$u(x_0, y_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x_0 + a\cos t, y_0 + a\sin t)\,dt.$$

Maximum/Minimum Principle

Theorem

Let Ω be an open, connected, and bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Assume that $u(x,y) \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ where $\overline{\Omega} = \Omega \cup \partial\Omega$, then

- *if $u_{xx} + u_{yy} \geq 0$ for all $(x,y) \in \Omega$, then*

$$\max_{(x,y)\in\overline{\Omega}} u(x,y) = \max_{(x,y)\in\partial\Omega} u(x,y),$$

- *if $u_{xx} + u_{yy} \leq 0$ for all $(x,y) \in \Omega$, then*

$$\min_{(x,y)\in\overline{\Omega}} u(x,y) = \min_{(x,y)\in\partial\Omega} u(x,y).$$

Maximum/Minimum Principle for Laplace's Equation

Corollary

Let Ω be an open, connected, and bounded domain in $\mathbb{R}^2$. Assume that $u(x,y) \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ where $\overline{\Omega} = \Omega \cup \partial\Omega$ and $u(x,y)$ satisfies Laplace's equation on Ω, then

$$\max_{(x,y)\in\overline{\Omega}} u(x,y) = \max_{(x,y)\in\partial\Omega} u(x,y)$$

and

$$\min_{(x,y)\in\overline{\Omega}} u(x,y) = \min_{(x,y)\in\partial\Omega} u(x,y).$$

Uniqueness of Solutions

Theorem (Uniqueness)

Suppose that Ω *is an open, connected, and bounded domain in* $\mathbb{R}^2$ *with* $f(x, y) \in \mathcal{C}(\Omega)$ *and* $\phi(x, y) \in \mathcal{C}(\partial\Omega)$*. If* $u(x, y) \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ *where* $\overline{\Omega} = \Omega \cup \partial\Omega$ *is a solution to the boundary value problem,*

$$\Delta u = f(x, y) \text{ for } (x, y) \in \Omega$$
$$u(x, y) = \phi(x, y) \text{ for } (x, y) \in \partial\Omega,$$

then this solution is unique.