Two-Point Boundary Value Problems

Partial Differential Equations

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Objectives

In this lesson we will learn:

- to classify second order, two-point boundary value problems,
- to place boundary value problems in self-adjoint form,
- properties of the eigenvalues and eigenfunctions of boundary value problems.

This enables us to generalize the method of separation of variables.

Background

When solving the heat equation for $0 \leq x \leq L$ with homogeneous boundary conditions we had to solve the induced **boundary value problem**:

$$\begin{aligned} X''(x) + \lambda\, X(x) &= 0 \\ X(0) &= 0 \\ X(L) &= 0. \end{aligned}$$

We saw that nontrivial solutions existed only if

$$\begin{aligned} \lambda = \lambda_n &= \frac{n^2\pi^2}{L^2} \\ X(x) = X_n(x) &= \sin\frac{n\pi x}{L} \text{ for } n \in \mathbb{N}. \end{aligned}$$

The set $\{\lambda_n\}_{n=1}^{\infty}$ is called the set of **eigenvalues** and $\{X_n(x)\}_{n=1}^{\infty}$ is the corresponding set of **eigenfunctions**.

Generalization

Consider the following second-order, linear homogeneous ODE of the form

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0 \text{ for } a < x < b.$$

Suppose that $P(x)$, $Q(x)$, and $R(x)$ are continuous on interval $[a, b]$ and that $P(x) \neq 0$ for all x in $[a, b]$.

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Suppose that $P(x)$, $Q(x)$, and $R(x)$ are continuous on interval $[a, b]$ and that $P(x) \neq 0$ for all x in $[a, b]$.

- If $Q(x) = P'(x)$ then the ODE can be written as

$$[P(x)y'(x)]' + R(x)y(x) = 0.$$

- If $Q(x) \neq P'(x)$ multiply both sides of the ODE by

$$r(x) = \frac{1}{P(x)} e^{\int_a^x Q(s)/P(s)\, ds} = \frac{p(x)}{P(x)}.$$

Self-Adjoint Form

$$
\begin{aligned}
P(x)y''(x) + Q(x)y'(x) + R(x)y(x) &= 0 \\
r(x)P(x)y''(x) + r(x)Q(x)y'(x) + r(x)R(x)y(x) &= 0 \\
p(x)y''(x) + p'(x)y'(x) + q(x)y(x) &= 0 \\
\left[p(x)y'(x)\right]' + q(x)y(x) &= 0
\end{aligned}
$$

where $q(x) = r(x)R(x)$.

An ODE written in this last form is called **self-adjoint**.

Example

Suppose $0 < a \leq x \leq b$ and express the following ODE in self-adjoint form.

$$x\,y''(x) + (1 - x)y'(x) + y(x) = 0$$

Example

Suppose $0 < a \leq x \leq b$ and express the following ODE in self-adjoint form.

$$x\, y''(x) + (1 - x)y'(x) + y(x) = 0$$

Note that

$$r(x) = \frac{1}{x} e^{\int_a^x (1-s)/s\, ds} = \frac{1}{a} e^{-x+a}.$$

Multiplying both sides of the ODE by $r(x)$ produces,

$$\begin{aligned} x\, y''(x) + (1 - x)y'(x) + y(x) &= 0 \\ \frac{x}{a} e^{-x+a}\, y''(x) + \frac{1}{a} e^{-x+a}(1 - x)y'(x) + \frac{1}{a} e^{-x+a} y(x) &= 0 \\ \left[x\, e^{-x+a} y'(x)\right]' + e^{-x+a} y(x) &= 0. \end{aligned}$$

Introduction of Operator Notation

Given that the ODE:

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$$

can be written in the self-adjoint form:

$$[p(x)y'(x)]' + q(x)y(x) = 0$$

define the **linear operator**,

$$L[y] = [p(x)y']' + q(x)y.$$

Sturm-Liouville BVP (1 of 2)

Consider a second-order linear ODE and boundary conditions:

$$[p(x)y']' + (q(x) + \lambda r(x))y = 0 \text{ for } a < x < b$$
$$\alpha_1 y(a) + \beta_1 y'(a) = 0$$
$$\alpha_2 y(b) + \beta_2 y'(b) = 0,$$

where $\alpha_1^2 + \beta_1^2 \neq 0$ and $\alpha_2^2 + \beta_2^2 \neq 0$.

Remarks:

- This is known as a **Sturm-Liouville BVP** with **separated boundary conditions**.
- The unknown constant λ is the **eigenvalue parameter**.
- The function $r(x)$ is called the **weight function**.
- The solution $y(x)$ is the **eigenfunction** corresponding to λ.

Sturm-Liouville BVP (2 of 2)

The boundary value problem can be concisely written as,

$$
\begin{aligned}
L[y] + \lambda r(x) y = 0 \text{ for } a < x < b \\
\alpha_1 y(a) + \beta_1 y'(a) = 0 \\
\alpha_2 y(b) + \beta_2 y'(b) = 0.
\end{aligned}
$$

Remarks: if

- $p(a) = 0$ or $p(b) = 0$, or
- $p(a) = \pm\infty$, $p(b) = \pm\infty$, $q(a) = \pm\infty$, $q(b) = \pm\infty$, $r(a) = \pm\infty$, or $r(b) = \pm\infty$, or
- $a = \pm\infty$ or $b = \pm\infty$, then

the BVP is **singular**. Otherwise the BVP is **regular**.

Example

Find the eigenvalues and eigenfunctions of the regular Sturm-Liouville BVP:

$$
\begin{aligned}
y''(x) + \frac{1}{x} y'(x) + \frac{\lambda}{x^2} y(x) &= 0 \text{ for } 1 < x < 2 \\
y(1) &= 0 \\
y(2) &= 0.
\end{aligned}
$$

Hint: make the change of variable $z = \ln x$.

Solution (1 of 5)

$$\frac{dy}{dx} = \frac{dy}{dz}\,\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} = e^{-z}\frac{dy}{dz}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^{-z}\frac{dy}{dz}\right] = e^{-2z}\frac{dy}{dz} + e^{-2z}\frac{d^2y}{dz^2}$$

Substituting into the BVP yields:

$$e^{-2z}\frac{d^2y}{dz^2} + e^{-2z}\frac{dy}{dz} + e^{-z}e^{-z}\frac{dy}{dz} + \lambda(e^{-z})^2 y = 0 \text{ for } 0 < z < \ln 2$$
$$y''(z) + 2y'(z) + \lambda y(z) = 0 \text{ for } 0 < z < \ln 2.$$

Solution (2 of 5)

If $\lambda = 1$ the general solution can be expressed as

$$y(z) = (C_1 + C_2 z)e^{-z} \implies y(x) = (C_1 + C_2 \ln x)\frac{1}{x}.$$

Using the boundary conditions:

$$y(1) = 0 = (C_1 + C_2 \ln 1)\frac{1}{1} \implies C_1 = 0$$
$$y(2) = 0 = \frac{C_2}{2} \ln 2 \implies C_2 = 0.$$

Solution (3 of 5)

If $\lambda < 1$ the general solution can be expressed as

$$y(z) = C_1 e^{(-1-(1-\lambda)^{1/2})z} + C_2 e^{(-1+(1-\lambda)^{1/2})z}$$
$$y(x) = C_1 x^{(-1-(1-\lambda)^{1/2})} + C_2 x^{(-1+(1-\lambda)^{1/2})}.$$

Using the boundary conditions:

$$y(1) = 0 = C_1 + C_2 \implies C_1 = -C_2$$
$$y(2) = 0 = C_1 2^{(-1-(1-\lambda)^{1/2})} - C_1 2^{(-1+(1-\lambda)^{1/2})} \implies C_1 = C_2 = 0.$$

Solution (4 of 5)

If $\lambda > 1$ the general solution can be expressed as

$$y(z) = C_1 e^{(-1-i(\lambda-1)^{1/2})z} + C_2 e^{(-1+i(\lambda-1)^{1/2})z}$$
$$y(x) = \frac{1}{x}\left(C_1 \cos((\lambda-1)^{1/2}\ln x) + C_2 \sin((\lambda-1)^{1/2}\ln x)\right).$$

Using the boundary conditions:

$$y(1) = 0 = C_1$$
$$y(2) = 0 = \frac{C_2}{2}\sin((\lambda-1)^{1/2}\ln 2).$$

- If $C_2 = 0$ then there are no nontrivial solutions.
- If $C_2 \neq 0$ then $(\lambda-1)^{1/2}\ln 2 = n\pi$ for $n \in \mathbb{N}$.

Solution (5 of 5)

Finally, the eigenvalues are

$$\lambda = \lambda_n = 1 + \frac{n^2\pi^2}{(\ln 2)^2}$$

with corresponding eigenfunctions

$$y(x) = y_n(x) = \frac{1}{x}\sin\left(\frac{n\pi}{\ln 2}\ln x\right)$$

for $n \in \mathbb{N}$.

Nontrivial Solutions (1 of 3)

Let $y_1(x)$ and $y_2(x)$ be any two linearly independent solutions to the BVP:

$$\begin{aligned} L[y] = [p(x)y']' + q(x)y = 0 \text{ for } a < x < b \\ \alpha_1 y(a) + \beta_1 y'(a) = 0 \\ \alpha_2 y(b) + \beta_2 y'(b) = 0, \end{aligned}$$

where $\alpha_1^2 + \beta_1^2 \neq 0$ and $\alpha_2^2 + \beta_2^2 \neq 0$.

If $y(x)$ is any nontrivial solution to the BVP then there exist constants C_1 and C_2 such that

$$y(x) = C_1 y_1(x) + C_2 y_2(x).$$

Nontrivial Solutions (2 of 3)

Since $y(x)$ satisfies the boundary conditions at $x = a$ and $x = b$, then

$$\begin{bmatrix} \alpha_1 y(a) + \beta_1 y'(a) \\ \alpha_2 y(b) + \beta_2 y'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 y_1(a) + \beta_1 y_1'(a) & \alpha_1 y_2(a) + \beta_1 y_2'(a) \\ \alpha_2 y_1(b) + \beta_2 y_1'(b) & \alpha_2 y_2(b) + \beta_2 y_2'(b) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrix equation has nontrivial solutions if and only if

$$\begin{vmatrix} \alpha_1 y_1(a) + \beta_1 y_1'(a) & \alpha_1 y_2(a) + \beta_1 y_2'(a) \\ \alpha_2 y_1(b) + \beta_2 y_1'(b) & \alpha_2 y_2(b) + \beta_2 y_2'(b) \end{vmatrix} = 0.$$

Nontrivial Solutions (3 of 3)

We can summarize these results as the following theorem.

Theorem

The BVP:

$$\begin{aligned}
&[p(x)y']' + q(x)y = 0 \text{ for } a < x < b \\
&\alpha_1 y(a) + \beta_1 y'(a) = 0 \\
&\alpha_2 y(b) + \beta_2 y'(b) = 0,
\end{aligned}$$

where $\alpha_1^2 + \beta_1^2 \neq 0$ *and* $\alpha_2^2 + \beta_2^2 \neq 0$ *has a nontrivial solution if and only if for any two linearly independent solutions* y_1 *and* y_2 *of the BVP,*

$$\begin{vmatrix} \alpha_1 y_1(a) + \beta_1 y_1'(a) & \alpha_1 y_2(a) + \beta_1 y_2'(a) \\ \alpha_2 y_1(b) + \beta_2 y_1'(b) & \alpha_2 y_2(b) + \beta_2 y_2'(b) \end{vmatrix} = 0.$$

In this case all solutions are $u(x) = C\,y(x)$ *where* y *is any nontrivial solution of the BVP and* C *is any constant.*

Example

Find a nontrivial solution to the BVP problem if a nontrivial solution exists.

$$x^2 y''(x) + x\,y(x) - 4y(x) = 0 \text{ for } 1 < x < 2$$
$$2y(1) + y'(1) = 0$$
$$y(2) + y'(2) = 0$$

Solution

The ODE is Euler's equation for which two linearly independent solutions are $y_1(x) = x^2$ and $y_2(x) = x^{-2}$.

Suppose $y(x) = C_1 x^2 + C_2 x^{-2}$ then to satisfy the boundary conditions:

$$0 = 2y(1) + y'(1) = 2C_1 + 2C_2 + 2C_1 - 2C_2 \implies C_1 = 0$$
$$0 = y(2) + y'(2) = \frac{C_2}{4} - \frac{2C_2}{8}$$

which is true for any choice of C_2.

Every nontrivial solution to the BVP has the form $y(x) = C/x^2$ where $C \neq 0$ is a constant.

Lagrange's Identity

Lemma (Lagrange's Identity)

Let L be the operator defined as

$$L[y] = [p(x)y']' + q(x)y$$

and let u and v be any twice continuously differentiable functions on (a, b), then

$$u\,L[v] - v\,L[u] = \frac{d}{dx}\left[p(x)\left(u(x)\frac{dv}{dx} - v(x)\frac{du}{dx}\right)\right].$$

Green's Formula

Lemma (Green's Formula)

Let L be the operator defined as

$$L[y] = [p(x)y']' + q(x)y$$

and let u and v be any twice continuously differentiable functions on (a, b), then

$$\int_a^b (uL[v] - vL[u])\ dx = \left[p(x)\left(u(x)\frac{dv}{dx} - v(x)\frac{du}{dx}\right)\right]_{x=a}^{x=b}.$$

Picone's Identity

Lemma (Picone's Identity)

Suppose that $u(x)$ and $v(x)$ are solutions to the following ordinary differential equations:

$$[p_1(x)u']' + q_1(x)u = 0$$
$$[p_2(x)v']' + q_2(x)v = 0.$$

For all x such that $v(x) \neq 0$, Picone's Identity holds,

$$\left[\frac{u}{v}(p_1 u' v - p_2 u v')\right]' = (q_2 - q_1)u^2 + (p_1 - p_2)(u')^2 + \frac{p_2(u'v - uv')^2}{v^2}.$$

Multiplicity of Eigenvalues

Consider the ODE:

$$L[y](x) + \lambda r(x)y(x) = [p(x)y'(x)]' + (q(x) + \lambda r(x))y(x) = 0.$$

Definition

An eigenvalue λ is said to be **simple** if there is only one linearly independent eigenfunction ϕ corresponding to λ. If there exist k linearly independent eigenfunctions corresponding to λ, the eigenvalue is said to be of **multiplicity** k.

Eigenvalues of Sturm-Liouville BVPs

Theorem

All eigenvalues of the regular Sturm-Liouville boundary value problem with separated boundary conditions:

$$
\begin{gathered}
[p(x)y'(x)]' + (q(x) + \lambda r(x))y(x) = 0 \text{ for } a < x < b \\
\alpha_1 y(a) + \beta_1 y'(a) = 0 \\
\alpha_2 y(b) + \beta_2 y'(b) = 0,
\end{gathered}
$$

with $\alpha_1^2 + \beta_1^2 \neq 0$ *and* $\alpha_2^2 + \beta_2^2 \neq 0$ *are simple.*

Inner Products and Weight Functions

Definition

Let $f(x)$ and $g(x)$ be two integrable functions on the interval $[a, b]$, and let $\rho(x)$ be a positive, continuous function on $[a, b]$. The **weighted inner product of f and g with respect to ρ on** $[a, b]$ denoted as $\langle \cdot, \cdot \rangle_\rho$ is

$$\langle f, g \rangle_\rho = \int_a^b f(x) g(x) \rho(x) \, dx.$$

Orthogonality

Theorem

Suppose $y_\lambda(x)$ and $y_\mu(x)$ are solutions to the BVP:

$$\begin{gathered}[p(x)y'(x)]' + (q(x) + \nu r(x))y(x) = 0 \text{ for } a < x < b \\ \alpha_1 y(a) + \beta_1 y'(a) = 0 \\ \alpha_2 y(b) + \beta_2 y'(b) = 0,\end{gathered}$$

corresponding to eigenvalues $\nu = \lambda$ and $\nu = \mu$ respectively. If $\lambda \neq \mu$, then $y_\lambda(x)$ and $y_\mu(x)$ are orthogonal with respect to the weight function $r(x)$.

Remark: eigenfunctions corresponding to distinct eigenvalues of a Sturm-Liouville boundary value problem are orthogonal.

Eigenvalues are Real

Theorem

The eigenvalues of the Sturm-Liouville boundary value problem:

$$[p(x)y'(x)]' + (q(x) + \nu r(x))y(x) = 0 \text{ for } a < x < b$$
$$\alpha_1 y(a) + \beta_1 y'(a) = 0$$
$$\alpha_2 y(b) + \beta_2 y'(b) = 0,$$

are all real and the corresponding eigenfunctions are real-valued functions (except possibly for a complex-valued constant multiplicative factor).