Existence of Eigenfunctions and Eigenvalues

Partial Differential Equations

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Objectives

In this lesson we will learn:

- to transform the regular Sturm-Liouville boundary value problem to an equivalent system of two first-order equations,
- to establish the existence of an infinite sequence of eigenfunctions, and
- to show the eigenvalues of the Sturm-Liouville boundary value problem form an increasing sequence which grows unbounded.

Sturm-Liouville Boundary Value Problem

Throughout this discussion we will consider the regular Sturm-Liouville BVP:

$$[p(x)y'(x)]' + (q(x) + \lambda r(x))y(x) = 0 \text{ for } a < x < b$$
$$\alpha_1 y(a) + \beta_1 y'(a) = 0$$
$$\alpha_2 y(b) + \beta_2 y'(b) = 0.$$

Pŕufer Substitution (1 of 2)

Convert the second-order ODE to a system of two first-order ODEs via the substitutions:

$$
\begin{aligned}
X(x) &= y(x) \\
Y(x) &= p(x)y'(x).
\end{aligned}
$$

This produces the first-order system:

$$
\begin{aligned}
X'(x) &= \frac{1}{p(x)}\,Y(x) \\
Y'(x) &= -(q(x) + \lambda\, r(x))X(x).
\end{aligned}
$$

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Now make the **Prüfer substitution**:

$$X(x) = R(x)\sin\theta(x)$$
$$Y(x) = R(x)\cos\theta(x).$$

Pr΄ufer Substitution (2 of 2)

The Sturm-Liouville ordinary differential equation is now written as

$$R'(x) = \frac{1}{2}\left(\frac{1}{p(x)} - [q(x) + \lambda\, r(x)]\right) R(x) \sin(2\theta(x))$$

$$\theta'(x) = \frac{1}{p(x)} \cos^2\theta(x) + [q(x) + \lambda\, r(x)] \sin^2\theta(x).$$

Remarks:

- Any solution to this system of ODEs will solve the Sturm-Liouville ODE provided the change of variables is invertible.
- The second equation is independent of R and is continuously differentiable with respect to θ. Therefore, for any initial value $\theta_0 \in [0, 2\pi)$, there exits a unique solution, denoted as $\theta_\lambda(x)$, such that $\theta_\lambda(a) = \theta_0$.

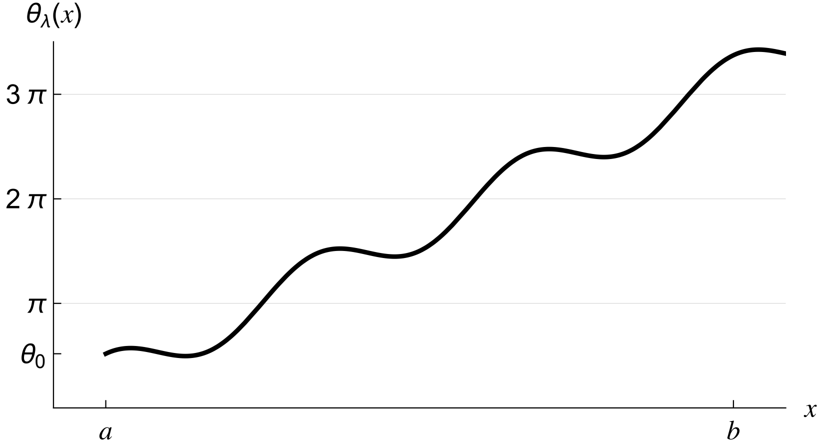
Properties of $\theta_\lambda(x)$

Lemma

Let λ be fixed. Suppose $z \in (a, b)$ is such that $\theta_\lambda(z) = k\pi$ for some $k \in \mathbb{Z}$, then z is the unique solution to $\theta_\lambda(x) = k\pi$ in (a, b). Furthermore $\theta_\lambda(x) < k\pi$ for $x < z$ and $\theta_\lambda(x) > k\pi$ for $x > z$.

Remark: the function $\theta_\lambda(x)$ is monotone increasing in an interval surrounding each multiple of π.

Illustration



Comparison of Solutions

Theorem

Let $p_i(x)$ be continuously differentiable on $[a, b]$ and let $q_i(x)$ be continuous on $[a, b]$ for $i = 1, 2$. Suppose $0 < p_2(x) \leq p_1(x)$ and $q_1(x) \leq q_2(x)$ on $[a, b]$. Let y_i be a solution of

$$[p_i(x)y'(x)]' + q_i(x)y(x) = 0,$$

and let $R_i(x)$ and $\theta_i(x)$ be the the corresponding solutions to the Prüfer form of the ordinary differential equation. If $\theta_1(a) \leq \theta_2(a)$, then $\theta_1(x) \leq \theta_2(x)$ for all $x \in [a, b]$. If in addition $q_1(x) < q_2(x)$ on $[a, b]$ then $\theta_1(x) < \theta_2(x)$ on $(a, b]$.

Transformed Boundary Conditions (1 of 2)

Apply the change of variables and Prüfer transformation to the boundary condition at $x = a$.

$$
\begin{aligned}
\alpha_1 y(a) + \beta_1 y'(a) &= 0 \\
\alpha_1 X(a) + \frac{\beta_1 Y(a)}{p(a)} &= 0 \\
\frac{\alpha_1 p(a)}{\sqrt{(\alpha_1 p(a))^2 + \beta_1^2}} X(a) + \frac{\beta_1}{\sqrt{(\alpha_1 p(a))^2 + \beta_1^2}} Y(a) &= 0
\end{aligned}
$$

Transformed Boundary Conditions (1 of 2)

Apply the change of variables and Prüfer transformation to the boundary condition at $x = a$.

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\alpha_1 y(a) + \beta_1 y'(a) &= 0 \\
\alpha_1 X(a) + \frac{\beta_1 Y(a)}{p(a)} &= 0 \\
\frac{\alpha_1 p(a)}{\sqrt{(\alpha_1 p(a))^2 + \beta_1^2}} X(a) + \frac{\beta_1}{\sqrt{(\alpha_1 p(a))^2 + \beta_1^2}} Y(a) &= 0
\end{aligned}$$

Without loss of generality, assume $\beta_1 > 0$. There exists a unique $A \in [0, \pi)$ such that

$$\cos A = \frac{\alpha_1 p(a)}{\sqrt{(\alpha_1 p(a))^2 + \beta_1^2}} \text{ and } \sin A = \frac{\beta_1}{\sqrt{(\alpha_1 p(a))^2 + \beta_1^2}}.$$

Transformed Boundary Conditions (2 of 2)

The boundary condition at $x = a$ can be written as

$$
\begin{aligned}
(\cos A)X(a) + (\sin A)Y(a) &= 0 \\
(\cos A)y(a) + (\sin A)p(a)y'(a) &= 0.
\end{aligned}
$$

Similarly there exists a unique $B \in (0, \pi]$ such that the boundary condition at $x = b$ is equivalent to

$$(\cos B)y(b) + (\sin B)p(b)y'(b) = 0.$$

Unboundedness of $\theta_\lambda(x)$

Lemma

Let $\phi_\lambda(x)$ be the unique solution to the regular Sturm-Liouville differential equation satisfying the initial conditions

$$\phi_\lambda(a) = \sin A \text{ and } \phi'_\lambda(a) = -\frac{\cos A}{p(a)}.$$

If $\theta_\lambda(x)$ is the solution to the Prüfer transformed ODE corresponding to $\phi_\lambda(x)$ then for any $z \in (a, b]$,

$$\lim_{\lambda\to\infty} \theta_\lambda(z) = \infty.$$

Another Limit of $\theta_\lambda(x)$

Lemma

Let $\phi_\lambda(x)$ be the unique solution to the regular Sturm-Liouville differential equation satisfying the initial conditions

$$\phi_\lambda(a) = \sin A \text{ and } \phi'_\lambda(a) = -\frac{\cos A}{p(a)}.$$

If $\theta_\lambda(x)$ is the solution to the Prüfer transformed ODE corresponding to $\phi_\lambda(x)$ then for any $z \in (a, b]$,

$$\lim_{\lambda \to -\infty} \theta_\lambda(z) = 0.$$

Main Result

Theorem

The eigenvalues and eigenfunctions of a regular Sturm-Liouville boundary value problem have the following properties.

(i) *The boundary value problem possesses an infinite sequence of eigenvalues* $\{\lambda_n\}_{n=1}^{\infty}$ *that can be arranged in increasing order as*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

with $\lim_{n\to\infty} \lambda_n = \infty$.

(ii) *Any eigenfunction* $\phi_n(x)$ *corresponding to eigenvalue* λ_n *has exactly* $n-1$ *zeroes in* $[a, b]$.

(iii) *The zeroes of* $\phi_{n+1}(x)$ *lie between the zeroes of* $\phi_n(x)$.

Example

Consider the boundary value problem:

$$
\begin{aligned}
X'' + \lambda X &= 0 \text{ for } 0 < x < 1 \\
X(0) - \beta_1 X'(0) &= 0 \\
X(1) + \beta_2 X'(1) &= 0,
\end{aligned}
$$

where β_1 and β_2 are positive constants. Show that the eigenfunctions and eigenvalues of this Sturm-Liouville BVP have the properties outlined in the previous theorem.

Solution (1 of 6)

Case: $\lambda = -k^2 < 0$

The general solution to the ODE is $X(x) = c_1 e^{kx} + c_2 e^{-kx}$. If $X(x)$ satisfies the boundary conditions then

$$\begin{aligned} c_1 + c_2 - \beta_1 k(c_1 - c_2) &= 0 \\ c_1 e^k + c_2 e^{-k} + \beta_2 k(c_1 e^k - c_2 e^{-k}) &= 0. \end{aligned}$$

The only solution to this system of equations is $c_1 = c_2 = 0$ and thus there are no nontrivial eigenfunctions when $\lambda < 0$.

Solution (2 of 6)

Case: $\lambda = 0$

The general solution to the ODE is $X(x) = c_1 x + c_2$. If $X(x)$ satisfies the boundary conditions then

$$\begin{aligned} c_2 - \beta_1 c_1 &= 0 \\ c_1 + c_2 + \beta_2 c_2 &= 0. \end{aligned}$$

The only solution to this system of equations is $c_1 = c_2 = 0$ and thus there are no nontrivial eigenfunctions when $\lambda = 0$.

Solution (3 of 6)

Case: $\lambda = k^2 > 0$

The general solution to the ODE is $X(x) = c_1 \cos(kx) + c_2 \sin(kx)$. If $X(x)$ satisfies the boundary conditions then

$$\begin{aligned} c_1 - \beta_1 k c_2 &= 0 \\ c_1 \cos(k) + c_2 \sin(k) + \beta_2 k(-c_1 \sin(k) + c_2 \cos(k)) &= 0. \end{aligned}$$

The first equation implies $c_1 = \beta_1 k c_2$. Substituting this into the second equation yields:

$$c_2 k(\beta_1 + \beta_2) \cos(k) + c_2(1 - k^2 \beta_1 \beta_2) \sin(k) = 0$$

If $c_2 = 0$ then $c_1 = 0$ and there are no nontrivial eigenfunctions.

Solution (4 of 6)

Assuming $c_2 \neq 0$ then

$$\begin{aligned} k(\beta_1 + \beta_2)\cos(k) + (1 - k^2\beta_1\beta_2)\sin(k) &= 0 \\ \frac{k(\beta_1 + \beta_2)}{k^2 - \beta_1\beta_2} &= \tan(k) \end{aligned}$$

provided $k \neq \dfrac{(2n-1)\pi}{2}$ for all $n \in \mathbb{N}$. This is equivalent to $\beta_1\beta_2 \neq \left(\dfrac{(2n-1)\pi}{2}\right)^2$ for all $n \in \mathbb{N}$.

Thus $\lambda = k^2 > 0$ is an eigenvalue of the BVP if and only if k solves the equation

$$\frac{k(\beta_1 + \beta_2)}{k^2 - \beta_1\beta_2} = \tan(k)$$

Solution (5 of 6)

There are infinitely many positive eigenvalues.

$$(n-1)^2\pi^2 < \lambda_n < \left(\frac{(2n-1)\pi}{2}\right)^2 \text{ for } n \in \mathbb{N}.$$

Numerically the first six eigenvalues are approximated as follows.

n	$\lambda_n = k_n^2$
1	1.707
2	13.492
3	43.357
4	92.769
5	161.881
6	250.719

The eigenfunction corresponding to eigenvalue λ_n is expressed as

$$X_n(x) = \cos(\sqrt{\lambda_n}x) + \frac{1}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}x).$$

Solution (6 of 6)