Generalized Fourier Series

Partial Differential Equations

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Fall 2025

Objectives

In this lesson we will learn to:

- represent suitable functions as series of eigenfunctions of Sturm-Liouville boundary value problems, and
- determine a formula for finding the coefficients of the series of eigenfunctions.

Sturm-Liouville BVP

Throughout this lesson assume $\{\lambda_n\}_{n=1}^{\infty}$ are the eigenvalues and $\{X_n(x)\}_{n=1}^{\infty}$ are the corresponding eigenfunctions of the following Sturm-Liouville boundary value problem.

$$[p(x)y'(x)]' + (q(x) + \lambda r(x))y(x) = 0 \text{ for } a < x < b$$
$$\alpha_1 y(a) + \beta_1 y'(a) = 0$$
$$\alpha_2 y(b) + \beta_2 y'(b) = 0$$

Recall that the inner product of functions f and g on $[a, b]$ with respect to weight function r is

$$\langle f, g \rangle_r = \int_a^b f(x)g(x)r(x)\,dx.$$

Representing a Piecewise Smooth Function

Question: if $f(x)$ is piecewise smooth on $[a, b]$ and $f(x) \sim \sum_{n=1}^{\infty} c_n X_n(x)$, what are the coefficients c_n for $n \in \mathbb{N}$?

Representing a Piecewise Smooth Function

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Answer: assuming

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x)$$

multiply both sides of the equation by $X_m(x)r(x)$ and integrate over $[a, b]$.

$$\int_a^b f(x) X_m(x) r(x)\, dx = \int_a^b \sum_{n=1}^{\infty} c_n X_m(x) X_n(x) r(x)\, dx$$

Generalized Fourier Coefficients

Assume the order of integration and summation can be interchanged on the right-hand side, then by the orthogonality of the eigenfunctions

$$\begin{aligned}\int_a^b f(x)X_m(x)r(x)\,dx &= \sum_{n=1}^{\infty} c_n \int_a^b X_m(x)X_n(x)r(x)\,dx \\ &= c_m \int_a^b (X_m(x))^2 r(x)\,dx \\ &= c_m \langle X_m, X_m\rangle_r\end{aligned}$$

and hence

$$c_m = \frac{\int_a^b f(x)X_m(x)r(x)\,dx}{\int_a^b (X_m(x))^2 r(x)\,dx} = \frac{\langle f, X_m\rangle_r}{\langle X_m, X_m\rangle_r}.$$

Normalized Eigenfunctions

Define

$$\phi_n(x) = \frac{X_n(x)}{\left(\int_a^b (X_n(x))^2 r(x)\,dx\right)^{1/2}} = \frac{X_n(x)}{\sqrt{\langle X_n, X_n \rangle_r}}$$

as the normalized eigenfunction corresponding to eigenvalue λ_n. The coefficient can be chosen as

$$c_n = \int_a^b f(x)\phi_n(x)r(x)\,dx = \langle f, \phi_n \rangle_r.$$

Main Result

Theorem

Assume that $\{\lambda_n\}_{n=1}^{\infty}$ *is the set of eigenvalues of a regular Sturm-Liouville boundary value problem. Let* $\{\phi_n(x)\}_{n=1}^{\infty}$ *be the corresponding normalized eigenfunctions. If* $f(x)$ *is piecewise smooth on interval* $[a, b]$*, the series*

$$\sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_n = \langle f, \phi_n \rangle_r$$

converges pointwise to $\frac{1}{2}\left[f(x+) + f(x-)\right]$ *for all* $x \in (a, b)$*. If* $f(x)$ *is continuous on* $[a, b]$ *and satisfies the boundary conditions of the Sturm-Liouville boundary value problem then the series converges uniformly and absolutely to* f *for all* $x \in [a, b]$*.*

Example

Consider the boundary value problem:

$$
\begin{aligned}
&X''(x) + \lambda X(x) = 0 \text{ for } 0 < x < 1 \\
&X(0) - \beta_1 X'(0) = 0 \\
&X(1) + \beta_2 X'(1) = 0,
\end{aligned}
$$

where β_1 and β_2 are positive constants. Determine the eigenvalues and normalized eigenfunctions of this Sturm-Liouville boundary value problem.

Solution (1 of 7)

Case: $\lambda = -k^2 < 0$

The general solution to the ODE is $X(x) = c_1 e^{kx} + c_2 e^{-kx}$. If $X(x)$ satisfies the boundary conditions then

$$\begin{aligned} c_1 + c_2 - \beta_1 k(c_1 - c_2) &= 0 \\ c_1 e^k + c_2 e^{-k} + \beta_2 k(c_1 e^k - c_2 e^{-k}) &= 0. \end{aligned}$$

The only solution to this system of equations is $c_1 = c_2 = 0$ and thus there are no nontrivial eigenfunctions when $\lambda < 0$.

Solution (2 of 7)

Case: $\lambda = 0$

The general solution to the ODE is $X(x) = c_1 x + c_2$. If $X(x)$ satisfies the boundary conditions then

$$\begin{aligned} c_2 - \beta_1 c_1 &= 0 \\ c_1 + c_2 + \beta_2 c_2 &= 0. \end{aligned}$$

The only solution to this system of equations is $c_1 = c_2 = 0$ and thus there are no nontrivial eigenfunctions when $\lambda = 0$.

Solution (3 of 7)

Case: $\lambda = k^2 > 0$

The general solution to the ODE is $X(x) = c_1 \cos(kx) + c_2 \sin(kx)$. If $X(x)$ satisfies the boundary conditions then

$$\begin{aligned} c_1 - \beta_1 k c_2 &= 0 \\ c_1 \cos(k) + c_2 \sin(k) + \beta_2 k(-c_1 \sin(k) + c_2 \cos(k)) &= 0. \end{aligned}$$

The first equation implies $c_1 = \beta_1 k c_2$. Substituting this into the second equation yields:

$$c_2 k(\beta_1 + \beta_2)\cos(k) + c_2(1 - k^2\beta_1\beta_2)\sin(k) = 0$$

If $c_2 = 0$ then $c_1 = 0$ and there are no nontrivial eigenfunctions.

Solution (4 of 7)

Assuming $c_2 \neq 0$ then

$$\begin{aligned} k(\beta_1 + \beta_2)\cos(k) + (1 - k^2\beta_1\beta_2)\sin(k) &= 0 \\ \frac{k(\beta_1 + \beta_2)}{k^2 - \beta_1\beta_2} &= \tan(k) \end{aligned}$$

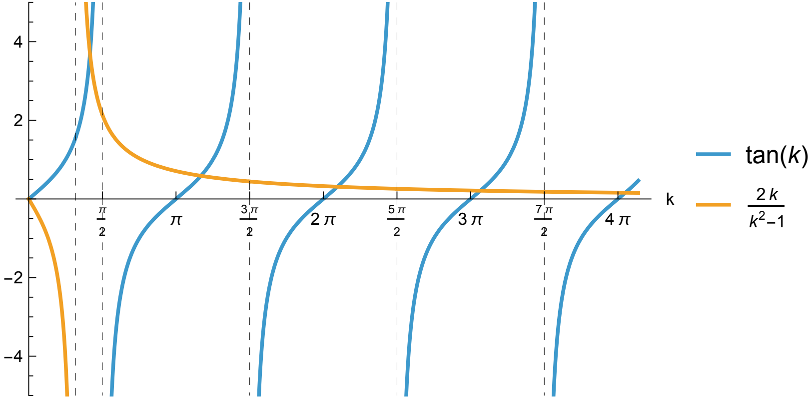
provided $k \neq \dfrac{(2n-1)\pi}{2}$ for all $n \in \mathbb{N}$. This is equivalent to $\beta_1\beta_2 \neq \left(\dfrac{(2n-1)\pi}{2}\right)^2$ for all $n \in \mathbb{N}$.

Thus $\lambda = k^2 > 0$ is an eigenvalue of the BVP if and only if k solves the equation

$$\frac{k(\beta_1 + \beta_2)}{k^2 - \beta_1\beta_2} = \tan(k)$$

Solution (5 of 7)

Suppose $\beta_1 = \beta_2 = 1$.



Solution (6 of 7)

There are infinitely many positive eigenvalues.

$$(n-1)^2\pi^2 < \lambda_n < \left(\frac{(2n-1)\pi}{2}\right)^2 \text{ for } n \in \mathbb{N}.$$

Numerically the first five eigenvalues are approximated as follows.

n	$\lambda_n = k_n^2$
1	13.492
2	43.357
3	92.769
4	161.881
5	250.719

The eigenfunction corresponding to eigenvalue λ_n is expressed as

$$X_n(x) = \cos(\sqrt{\lambda_n}x) + \frac{1}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}x).$$

Solution (7 of 7)

The normalized eigenfunctions can be expressed as

$$\phi_n(x) = \frac{X_n(x)}{\sqrt{\left\langle \int_0^1 (X_n(x))^2 \, dx \right\rangle}}.$$