Rayleigh Quotient

Partial Differential Equations

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Objectives

In this lesson we will learn:

- the formula for the Rayleigh quotient, and
- use the Rayleigh quotient to estimate the eigenvalues of a Sturm-Liouville boundary value problem.

Sturm-Liouville BVP

Once again, the setting of this discussion is the regular Sturm-Liouville boundary value problem:

$$
\begin{gathered}
[p(x)y'(x)]' + (q(x) + \lambda r(x))y(x) = 0 \text{ for } a < x < b \\
\alpha_1 y(a) + \beta_1 y'(a) = 0 \\
\alpha_2 y(b) + \beta_2 y'(b) = 0
\end{gathered}
$$

where $\alpha_1^2 + \beta_1^2 \neq 0$ and $\alpha_2^2 + \beta_2^2 \neq 0$.

Rayleigh Quotient Formula

Multiply both sides of the ODE by an eigenfunction $\phi(x)$ and integrate over $[a, b]$.

$$\begin{aligned} [p(x)\phi'(x)]' + (q(x) + \lambda r(x))\phi(x) &= 0 \\ [p(x)\phi'(x)]'\phi(x) + (q(x) + \lambda r(x))(\phi(x))^2 &= 0 \\ \int_a^b [p(x)\phi'(x)]'\phi(x) + (q(x) + \lambda r(x))(\phi(x))^2 \, dx &= 0 \end{aligned}$$

Expand the integrand and solve for eigenvalue λ.

Eigenvalue

$$
\begin{aligned}
0 &= \int_a^b [p(x)\phi'(x)]'\phi(x) + (q(x) + \lambda r(x))(\phi(x))^2\, dx \\
&= \int_a^b [p(x)\phi'(x)]'\phi(x)\, dx + \int_a^b q(x)(\phi(x))^2\, dx + \lambda \int_a^b r(x)(\phi(x))^2\, dx \\
\lambda \int_a^b r(x)(\phi(x))^2\, dx &= -\int_a^b [p(x)\phi'(x)]'\phi(x)\, dx - \int_a^b q(x)(\phi(x))^2\, dx \\
&= -\left[p(x)\phi(x)\phi'(x)\right]_{x=a}^{x=b} + \int_a^b p(x)(\phi'(x))^2\, dx - \int_a^b q(x)(\phi(x))^2\, dx \\
\lambda &= \frac{[-p(x)\phi(x)\phi'(x)]_{x=a}^{x=b} + \int_a^b \left(p(x)(\phi'(x))^2 - q(x)(\phi(x)^2)\right)\, dx}{\int_a^b r(x)(\phi(x))^2\, dx}
\end{aligned}
$$

Remarks:

- Note the use of integration by parts.
- This is not an effective way to calculate the eigenvalue λ since we must know the eigenfunction ϕ.

Positivity of Eigenvalues

Theorem

All eigenvalues of the Sturm-Liouville boundary value problem are nonnegative provided $q(x) \leq 0$ on $[a, b]$ and

$$p(b)\phi(b)\frac{d\phi}{dx}(b) - p(a)\phi(a)\frac{d\phi}{dx}(a) \leq 0.$$

$$\lambda = \frac{\left[-p(x)\phi(x)\phi'(x)\right]_{x=a}^{x=b} + \int_a^b \left(p(x)(\phi'(x))^2 - q(x)(\phi(x))^2\right)\,dx}{\int_a^b r(x)(\phi(x))^2\,dx}$$

Rayleigh Quotient Functional

Define the Rayleigh quotient functional as

$$R[y] = \frac{-\int_a^b y(x)([p(x)y'(x)]' + q(x)y(x))\,dx}{\int_a^b r(x)(y(x))^2\,dx}.$$

Note if $\phi_n(x)$ is the eigenfunction corresponding to eigenvalue λ_n then $R[\phi_n] = \lambda_n$.

Later we will see that every Sturm-Liouville BVP has a smallest eigenvalue λ_1.

The functional $R[\cdot]$ can be applied to functions $y(x)$ that are not eigenfunctions. Such a function $y(x)$ will be called a **trial function**.

Result

Theorem

Suppose $y(x)$ is piecewise smooth on $[a, b]$ such that $R[y]$ is well-defined and $y(x)$ satisfies the boundary conditions of the Sturm-Liouville BVP. Then $R[y] \geq \lambda_1$ where λ_1 is the smallest eigenvalue the Sturm-Liouville BVP. Equality is achieved only when $y(x) = \phi_1(x)$, the eigenfunction corresponding to λ_1.

Example

Consider the following boundary value problem:

$$y'' + \lambda y = 0 \text{ for } 0 < x < 1$$
$$y'(0) = y'(1) = 0.$$

Use trial functions to estimate the smallest eigenvalue of this problem.

Solution (1 of)

Note that $[a,b]=[0,1]$, $p(x)=1$, $r(x)=1$, $q(x)=0$, and for any function satisfying these boundary conditions the Rayleigh quotient simplifies to

$$R[y]=\frac{\int_0^1 (y'(x))^2\,dx}{\int_0^1 (y(x))^2\,dx}.$$

One trial function is $y_1(x)=(x-1)^2+\frac{2}{3}(x-1)^3$

Applying the Rayleigh quotient produces

$$R[y_1]=\frac{\int_0^1 \left(2(x-1)+2(x-1)^2\right)^2\,dx}{\int_0^1 \left((x-1)^2+\frac{2}{3}(x-1)^3\right)^2\,dx}=\frac{2/15}{13/315}=\frac{42}{13}\approx 3.23077.$$

Hence we know the smallest eigenvalue $\lambda_1\leq\frac{42}{13}$.

Solution (2 of)

Another trial function is $y_2(x) = \cos \pi x$.

$$R[y_2] = \frac{\int_0^1 (-\pi \sin \pi x)^2 \, dx}{\int_0^1 (\cos \pi x)^2 \, dx} = \frac{\pi^2/2}{1/2} = \pi^2 \approx 9.86960$$

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Remark: $y_2(x)$ is an eigenfunction of the boundary value problem corresponding to $\lambda = \pi^2$.

Solution (3 of)

A third trial function is $y_3(x) = 1$.

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Remark: $y_3(x)$ is actually the first eigenfunction for the boundary value problem and corresponds to the smallest eigenvalue $\lambda_1 = 0$.

Additional Remarks

- Minimizing the Rayleigh quotient over the appropriate vector space of smooth functions $y(x)$ defined on $[a, b]$ yields λ_1 and $\phi_1(x)$.
- Repeating the process of minimizing $R[y]$ over the functions orthogonal to $\phi_1(x)$ produces λ_2 and $\phi_2(x)$.
- This process can be iterated to yield numerical estimates of additional eigenvalues and eigenfunctions for the Sturm-Liouville boundary value problem.