Zeros of Eigenfunctions

Partial Differential Equations

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Objectives

In this lesson we will learn:

- about the placement of zeros of eigenfunctions,
- the relationship between zeros of different eigenfunctions,
- the relationship between zeros of eigenfunction to different regular Sturm-Liouville boundary value problems.

Isolated Zeros

Consider the Sturm-Liouville BVP:

$$[p(x)y']' + (q(x) + \lambda r(x))y = 0 \text{ for } a < x < b$$
$$\alpha_1 y(a) + \beta_1 y'(a) = 0$$
$$\alpha_2 y(b) + \beta_2 y'(b) = 0$$

with $\alpha_1^2 + \beta_1^2 > 0$ and $\alpha_2^2 + \beta_2^2 > 0$.

Theorem

Let $\phi(x) \in \mathcal{C}^1[a, b]$ be a nontrivial solution to the regular Sturm-Liouville BVP. If $\phi(x_0) = 0$ for some $x_0 \in (a, b)$ then there is an open interval $(x_0 - \delta, x_0 + \delta)$ with $\delta > 0$ which contains no other zero of $\phi(x)$.

Example

While not a proof of the previous theorem, the following example exhibits the behavior of the eigenfunctions described.

Find the eigenfunctions and their zeros for the following Sturm-Liouville BVP. Show that the zeros of the eigenfunctions are isolated,

$$
\begin{aligned}
x^2 y'' + xy' + \lambda y &= 0 \text{ for } 1 < x < 2 \\
y(1) &= 0 \\
y(2) &= 0
\end{aligned}
$$

Solution (1 of 5)

This is an example of Euler's differential equation. Make the change of variable $x = e^z \iff z = \ln x$. This produces the boundary value problem:

$$\begin{aligned} \frac{d^2y}{dz^2} + \lambda y &= 0 \\ y(0) &= 0 \\ y(\ln 2) &= 0. \end{aligned}$$

Solution (2 of 5)

Case: $\lambda = -k^2 < 0$

The general solution is $y(z) = c_1 e^{kz} + c_2 e^{-kz}$ where c_1 and c_2 are constants. The boundary conditions imply,

$$y(0) = c_1 + c_2 = 0$$
$$y(\ln 2) = 2^k c_1 + \frac{1}{2^k} c_2 = 0.$$

The only solution is $c_1 = c_2 = 0$ and thus there are no nontrivial solutions (eigenfunctions) when $\lambda < 0$.

Solution (3 of 5)

Case: $\lambda = 0$

The general solution is $y(z) = c_1 z + c_2$ where c_1 and c_2 are constants. The boundary conditions imply,

$$\begin{aligned} y(0) &= c_2 = 0 \\ y(\ln 2) &= (\ln 2)c_1 + c_2 = 0. \end{aligned}$$

The only solution is $c_1 = c_2 = 0$ and thus there are no nontrivial solutions (eigenfunctions) when $\lambda = 0$.

Solution (4 of 5)

Case: $\lambda = k^2 > 0$

The general solution is $y(z) = c_1 \cos(kz) + c_2 \sin(kz)$ where c_1 and c_2 are constants. The boundary conditions imply,

$$\begin{aligned} y(0) &= c_1 = 0 \\ y(\ln 2) &= c_1 \cos(k \ln 2) + c_2 \sin(k \ln 2) = 0. \end{aligned}$$

Certainly $c_1 = 0$. If $c_2 = 0$ then once again there are no nontrivial solutions. However if

$$k \ln 2 = n\pi \iff k = \frac{n\pi}{\ln 2}$$

for $n \in \mathbb{N}$ then

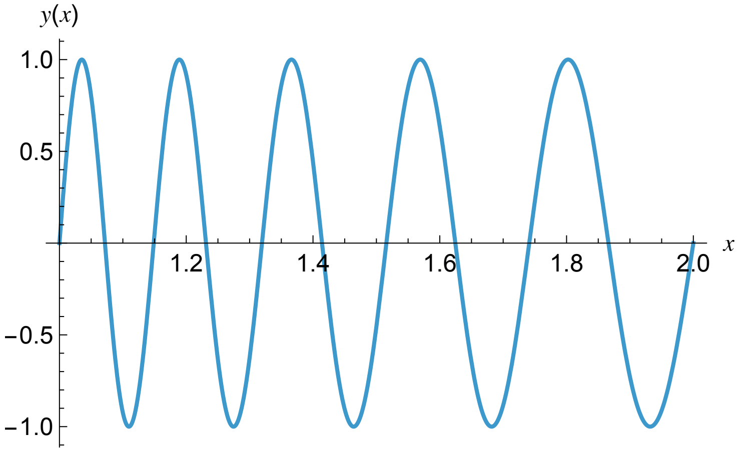
$$y(z) = c_2 \sin\left(\frac{n\pi z}{\ln 2}\right) \implies y(x) = c_2 \sin\left(\frac{n\pi \ln x}{\ln 2}\right)$$

is a nontrivial solution (eigenfunction) for any $c_2 \neq 0$. The corresponding eigenvalues are

$$\lambda = \lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2.$$

Solution (5 of 5)

The zeros of the eigenvalues are isolated from one another in the interval $[1, 2]$. For instance if $n = 10$ the graph of $y_{10}(x)$ resembles the following.



Separation of Zeros

Theorem (Sturm Separation Theorem)

Suppose $\phi(x)$ *and* $\psi(x)$ *are linearly independent solutions to the regular Sturm-Liouville BVP, then* $\psi(x)$ *has a zero strictly between any two zeros of* $\phi(x)$*.*

Example

Find two linearly independent solutions y_1 and y_2 to the following Sturm-Liouville BVP and show that between any two zeros of y_1 there is a zero of y_2.

$$\begin{aligned} x^2y'' + xy' + \lambda y &= 0 \text{ for } 1 < x < 2 \\ y(1) &= 0 \\ y(2) &= 0 \end{aligned}$$

Solution (1 of 2)

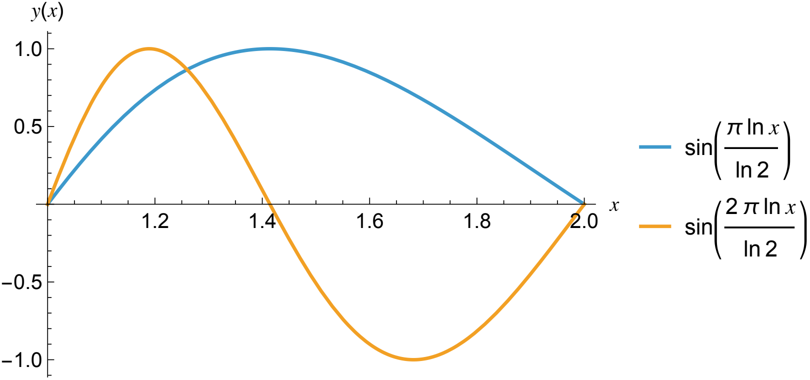
In the previous example we found that $y(x) = \sin\left(\frac{n\pi \ln x}{\ln 2}\right)$ is a nontrivial solution to the BVP. Let

$$y_1(x) = \sin\left(\frac{\pi \ln x}{\ln 2}\right)$$
$$y_2(x) = \sin\left(\frac{2\pi \ln x}{\ln 2}\right).$$

Note that $y_1(x) = 0$ only for $x = 1$ and $x = 2$.

The eigenfunction $y_2(x) = 0$ for $1 < x = \sqrt{2} < 2$.

Solution (2 of 2)



Interlacing of Zeros

Theorem (Sturm Comparison Theorem)

Suppose $\phi(x)$ and $\psi(x)$ are nontrivial solutions to the ordinary differential equations:

$$[p_1(x)\phi'(x)]'(x) + (q_1(x) + \lambda_1 r_1(x))\phi(x) = 0$$
$$[p_2(x)\psi'(x)]'(x) + (q_2(x) + \lambda_2 r_2(x))\psi(x) = 0$$

on interval (a, b). Suppose further that $0 < p_2(x) \leq p_1(x)$ and $q_1(x) + \lambda_1 r_1(x) \leq q_2(x) + \lambda_2 r_2(x)$ for all $x \in [a, b]$. If $\phi(x)$ has zeros at $x = \alpha$ and $x = \beta$ with $\alpha < \beta$, then $\psi(x)$ must have a zero in $[\alpha, \beta]$.

Example

Consider the two boundary value problems:

$$y'' + (1 + \lambda_1)y = 0 \qquad\qquad y'' + (2 + \lambda_2)y = 0$$

for $0 < x < 1$ subject to the boundary conditions:

$$y(0) = 0$$
$$y(1) = 0.$$

Find the eigenfunctions for each boundary value problem and show that their zeros satisfy the Sturm Comparison Theorem.

Solution (1 of 2)

We can check that $\phi(x) = \sin(m\pi x)$ solves the first BVP when $1 + \lambda_1 = m^2\pi^2$ while $\psi(x) = \sin(n\pi x)$ when $1 + \lambda_2 = n^2\pi^2$ solves the second.

Note that the leading coefficient in each BPV is $p_1(x) = p_2(x) = 1$. If $1 + \lambda_2 > 1 + \lambda_1$ or equivalently if $n > m$ then between two consecutive zeros of $\phi(x)$ there must be a zero of $\psi(x)$.

Solution (2 of 2)