Chebyshev Polynomials

Partial Differential Equations

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Objectives

In this lesson we will:

- explore the Chebyshev ordinary differential equation,
- define the Chebyshev polynomials, and
- describe the properties of the Chebyshev polynomials,

Chebyshev Ordinary Differential Equation

The **Chebyshev differential equation** is

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0 \text{ for } -1 < x < 1,$$

where α is a constant. The solutions of this equation have many applications in the physical sciences and in numerical mathematics.

Two linearly independent solutions to this ODE are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{\prod_{k=0}^{n-1}((2k)^2 - \alpha^2)}{(2n)!} x^{2n}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{\prod_{k=0}^{n-1}((2k+1)^2 - \alpha^2)}{(2n+1)!} x^{2n+1}.$$

Chebyshev Polynomials

If $\alpha = 2N$ where N is a nonnegative integer, the solution reduces to the polynomial,

$$y_1(x) = 1 + \sum_{n=1}^{N} (-1)^n \frac{\prod_{k=0}^{n-1}(N+k)(N-k)}{(2n)!}(2x)^{2n}.$$

If $\alpha = 2N + 1$ for some nonnegative integer N, the solution becomes the polynomial,

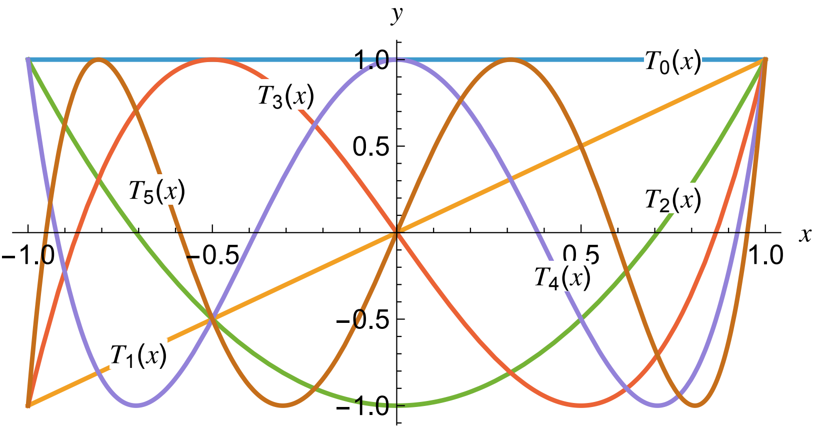
$$y_2(x) = x + \frac{1}{2}\sum_{n=1}^{N} (-1)^n \frac{\prod_{k=0}^{n-1}(N+k+1)(N-k)}{(2n+1)!}(2x)^{2n+1}.$$

By convention these polynomials are denoted $T_n(x)$ and are called the **Chebyshev polynomials of the first kind**.

Examples

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x
\end{aligned}
$$

Illustration



Properties (1 of 3)

Lemma

For each nonnegative integer n,

$$T_n(\cos\theta) = \cos(n\theta).$$

As a consequence, $T_n(x) = \cos(n \arccos x)$.

Recurrence relation:

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$$

Properties (2 of 3)

Theorem

The polynomial $T_n(x)$ has n distinct zeros in the interval $(-1, 1)$ located at

$$x_k = \cos\frac{(2k-1)\pi}{2n} \text{ for } k = 1, 2, \ldots, n.$$

Theorem

For $n \in \mathbb{N}$ the leading coefficient of $T_n(x)$ is 2^{n-1}.

Theorem

For $n \in \mathbb{N}$ the nth Chebyshev polynomial is given by the formula

$$T_n(x) = 2^{n-1}\prod_{k=1}^{n}\left(x - \cos\frac{(2k-1)\pi}{2n}\right).$$

Properties (3 of 3)

Theorem

For $m,\ n \in \{0, 1, 2, \ldots\}$ the Chebyshev polynomials are orthogonal on $[-1, 1]$ with respect to the weighting function $(1 - x^2)^{-1/2}$ and

$$\int_{-1}^{1} \frac{T_m(x) T_n(x)}{\sqrt{1 - x^2}}\, dx = \begin{cases} \pi & \text{if } m = n = 0, \\ \frac{\pi}{2} \delta_{mn} & \text{otherwise,} \end{cases}$$

where δ_{mn} is the Kronecker delta function.

Theorem

For $n \in \{0, 1, 2, \ldots\}$ the Chebyshev polynomial of degree n, $T_n(x)$ can be expressed as

$$T_n(x) = \frac{(-2)^n n!}{(2n)!} (1 - x^2)^{\frac{1}{2}} \frac{d^n}{dx^n} \left[(1 - x^2)^{n - \frac{1}{2}} \right].$$

Generalized Fourier Series

If $f(x)$ is sufficiently smooth on $(-1, 1)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n T_n(x),$$

where

$$a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_n(x)}{\sqrt{1 - x^2}}\, dx,$$

for $n = 0, 1, 2, \ldots$.