Gamma and Bessel Functions

Partial Differential Equations

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Objectives

In this lesson we will learn:

- the definition of the Gamma function,
- the properties of the Gamma function,
- how to solve Bessel's ordinary differential equation of order p,
- notable Bessel's functions, and
- properties of Bessel's functions.

Introducing the Gamma Function

The **Gamma function** is denoted as $\Gamma(p)$ and is defined by the integral,

$$\Gamma(p+1) = \int_0^\infty e^{-x} x^p \, dx.$$

Remarks:

- The improper integral converges for all $p \geq 0$.
- For $p < 0$ the integrand becomes unbounded as $x \to 0^+$.
- The integral can be shown to converge for $p > -1$ and to diverge for $p \leq -1$.

Example

Consider the case when $p = 0$,

$$\Gamma(1) = \int_0^{\infty} e^{-x}\, dx = \lim_{M\to\infty} \int_0^{M} e^{-x}\, dx = \lim_{M\to\infty} \left(-e^{-M} + 1\right) = 1.$$

Connection with Factorial

Lemma

For $p > 0$,

$$\Gamma(p+1) = p\,\Gamma(p).$$

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Lemma

If p is a positive integer,

$$\Gamma(p+1) = p!.$$

Remark: since the Gamma function is defined for non-integer values of p, it generalizes the factorial.

Example: $\Gamma(1/2)$

By definition,

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x} x^{-\frac{1}{2}}\, dx.$$

Making the substitution $u^2 = x$ and $2u\, du = dx$, the integral can be re-written as

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-u^2} u^{-1}(2u)\, du = 2\int_0^\infty e^{-u^2}\, du = \int_{-\infty}^\infty e^{-u^2}\, du = \sqrt{\pi}.$$

Pochhammer Symbol

Lemma

For $p > 0$,

$$\frac{\Gamma(p+n)}{\Gamma(p)} = p(p+1)(p+2)\cdots(p+n-1) = (p)_n.$$

Remark: the expression $(p)_n$ is called the **Pochhammer symbol** or the **rising factorial**.

Pochhammer Symbol

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Remark: the expression $(p)_n$ is called the **Pochhammer symbol** or the **rising factorial**.

This lemma makes it possible to calculate $\Gamma(p)$ for all $p > 0$ provided $\Gamma(x)$ is known for $0 < x < 1$.

Examples

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{11}{2}\right) = \Gamma\left(5+\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)\left(\frac{9}{2}\right) = \frac{945\sqrt{\pi}}{32}$$

Properties of the Gamma Function (1 of 2)

Lemma

The Gamma function $\Gamma(x)$ is continuous for all $x > 0$.

Properties of the Gamma Function (1 of 2)

Lemma

The Gamma function $\Gamma(x)$ *is continuous for all* $x > 0$.

The Gamma function is differentiable for $x > 0$.

$$\Gamma'(x) = \frac{d}{dx}\int_0^\infty e^{-t}t^{x-1}\,dt = \int_0^\infty \frac{d}{dx}\left[e^{-t}t^{x-1}\right]\,dt = \int_0^\infty e^{-t}(\ln t)t^{x-1}\,dt$$

Properties of the Gamma Function (1 of 2)

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The Gamma function to be extended to the real numbers in the set $\{x \in \mathbb{R} \,|\, x \neq 0, -1, -2, \ldots\}$. If n is a natural number and if $-n < x < -n + 1$ then $0 < x + n < 1$ and thus $\Gamma(x + n)$ is already defined. Recall that

$$\Gamma(x + n) = \Gamma(x)(x)_n$$
$$\Gamma(x) = \frac{\Gamma(x + n)}{(x)_n}.$$

This extended form of the Gamma function is also continuous on the set $\{x \in \mathbb{R} \,|\, x \neq 0, -1, -2, \ldots\}$.

Properties of the Gamma Function (2 of 2)

Theorem

If $n = 0, -1, -2, \ldots$ *then* $\lim_{x \to n} |\Gamma(x)| = \infty$.

Properties of the Gamma Function (2 of 2)

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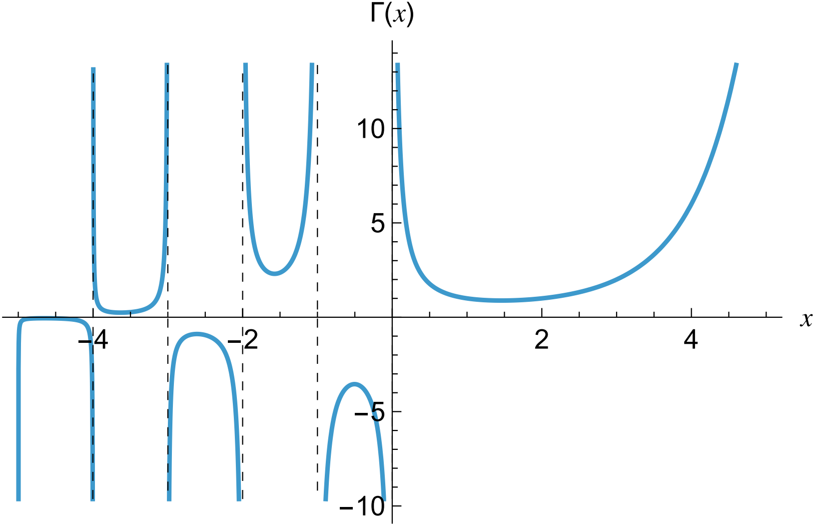
If $n = 0, -1, -2, \ldots$ *then* $\lim_{x \to n} |\Gamma(x)| = \infty$*.*

- Define the reciprocal of the Gamma function at $x = 0, -1, -2, \ldots$ to be 0. Thus $1/\Gamma(x)$ is continuous for all x and $1/\Gamma(x) = 0$ at $x = 0, -1, -2, \ldots$.
- Define $f(x) = \ln |\Gamma(x)|$ then

$$f'(x) = \frac{\Gamma'(x)}{\Gamma(x)} \iff \Gamma'(x) = \Gamma(x) f'(x).$$

- Function $f'(x)$ is called the **digamma function** or **polygamma function** and it is denoted as $\psi(x)$.
- Adopting this notation allows the derivative of the Gamma function to be written as $\Gamma'(x) = \psi(x)\Gamma(x)$.

Illustration of the Gamma Function



Bessel's Equation of Order $p \geq 0$

$$x^2 y'' + xy' + (x^2 - p^2)y = 0 \text{ for } x > 0,$$

The self-adjoint form is

$$[x\,y']' + \left(x - \frac{p^2}{x}\right) y = 0 \text{ for } x > 0.$$

The method of Frobenius will be used to solve this ODE.

The value $x_0 = 0$ is a regular singular point with indicial function $F(r) = r^2 - p^2$, and exponents of singularity $\pm p$.

Solving Bessel's Equation (1 of 4)

Assume $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$, differentiate formally with respect to x, substitute the result into Bessel's equation and re-index and combine the series to yield

$$\begin{aligned}
0 &= x^2 \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2} + x \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1} \\
&\quad + (x^2 - p^2) \sum_{k=0}^{\infty} a_k x^{k+r} \\
&= \sum_{k=0}^{\infty} [(k+r)(k+r-1) + (k+r) - p^2] a_k x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r+2} \\
&= \sum_{k=0}^{\infty} [(k+r)^2 - p^2] a_k x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r+2} \\
&= \sum_{k=0}^{\infty} F(r+k) a_k x^{k+r} + \sum_{k=2}^{\infty} a_{k-2} x^{k+r} \\
&= F(r) a_0 x^r + F(r+1) a_1 x^{r+1} + \sum_{k=2}^{\infty} [F(r+k) a_k + a_{k-2}] x^{k+r}.
\end{aligned}$$

Solving Bessel's Equation (2 of 4)

$$0 = F(r)a_0x^r + F(r+1)a_1x^{r+1} + \sum_{k=2}^{\infty}[F(r+k)a_k + a_{k-2}]x^{k+r}$$

- Since $r = -p$ or $r = p$ then $F(r) = 0$ and for arbitrary a_0 the first term above is zero.
- The second term has coefficient $a_1F(r+1) = a_1((1+r)^2 - p^2)$ which must match the coefficient of x^{r+1} on the left-hand side of the equation. This can be achieved by choosing $a_1 = 0$ regardless of the values of p or r.
- For $k \geq 2$ the following recurrence relation holds for any r such that $F(k+r) \neq 0$,

$$a_k = -\frac{a_{k-2}}{F(k+r)}.$$

Solving Bessel's Equation (3 of 4)

$$a_k = -\frac{a_{k-2}}{F(k+r)}$$

This equation is valid as long as r is the larger of the two exponents of singularity ($r = p$). Since $a_1 = 0$ then $a_{2k-1} = 0$ for $k \in \mathbb{N}$. Since $F(k+p) \neq 0$ for $k \in \mathbb{N}$, the recurrence relation can be simplified to

$$a_k = -\frac{a_{k-2}}{k(k+2p)}.$$

Since a_0 is an arbitrary constant, then

$$\begin{aligned}
a_2 &= -\frac{a_0}{2(2+2p)} = \frac{(-1)^1 a_0}{2^2(1+p)} \\
a_4 &= -\frac{a_2}{4(4+2n)} = \frac{(-1)^2 a_0}{2^4 2!(1+p)(2+p)} \\
&\vdots \\
a_{2k} &= \frac{(-1)^k a_0}{2^{2k} k!(1+p)(2+p)\cdots(k+p)}.
\end{aligned}$$

Solving Bessel's Equation (4 of 4)

One solution to Bessel's equation of order p has the form,

$$y_1(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k!(1+p)(2+p)\cdots(k+p)} x^{2k+p}.$$

Traditionally the first solution to Bessel's equation of order p has the arbitrary constant a_0 chosen as

$$a_0 = \frac{1}{2^p \Gamma(p+1)}.$$

This defines the **Bessel function of the first kind of order** p:

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p}.$$

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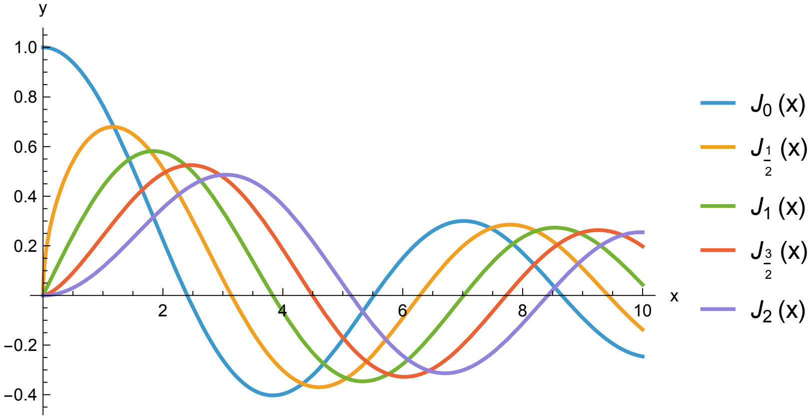
When $p = n \in \{0, 1, \ldots\}$ then $J_n(x)$ is a power series,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}.$$

Bessel Functions of the First Kind



Second Linearly Independent Solution (1 of 3)

Since Bessel's equation is second order, there must be another solution that is linearly independent of $J_p(x)$.

If $2p$ is not an integer then a second solution is

$$J_{-p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-p+1)} \left(\frac{x}{2}\right)^{2k-p}.$$

By convention the second solution to Bessel's equation of order p, linearly independent of $J_p(x)$ is chosen to be a linear combination of $J_p(x)$ and $J_{-p}(x)$, namely

$$Y_p(x) = \frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)}.$$

The function $Y_p(x)$ is called a **Bessel function of the second kind of order** p.

Second Linearly Independent Solution (2 of 3)

If $p = n \in \mathbb{N}$ then $2p$ is an integer. Assume a second solution to Bessel's equation if order n, linearly independent from $J_n(x)$ takes the form

$$y_2(x) = aJ_n(x) \ln x + x^{-n} \left(1 + \sum_{k=1}^{\infty} c_k x^k \right)$$

where a and c_k for $k \in \mathbb{N}$ are constants.

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where a and c_k for $k \in \mathbb{N}$ are constants.

It can be shown that

$$y_2(x) = \frac{-\ln x}{2^{n-1}(n-1)!} J_n(x) + x^{-n} \left[1 + \sum_{k=1}^{n-1} \frac{(n-k-1)!}{k!(n-1)!} \left(\frac{x}{2} \right)^{2k} \right] + x^{-n} \left[\sum_{k=0}^{\infty} \frac{(-1)^k (H_k + H_{n+k})}{k!(n+k)!(n-1)!} \left(\frac{x}{2} \right)^{2n+2k} \right],$$

where $H_k = \sum_{j=1}^{k} \frac{1}{j}$ is the kth Harmonic number.

Second Linearly Independent Solution (3 of 3)

By convention the second linearly independent solution to Bessel's equation of order $n \in \mathbb{N}$ is expressed as

$$\begin{aligned} Y_n(x) &= \frac{2}{\pi}\left((\gamma - \ln 2)J_n(x) - 2^{n-1}(n-1)!y_2(x)\right) \\ &= \frac{2}{\pi}\left(\gamma + \ln\frac{x}{2}\right)J_n(x) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n} \\ &\quad - \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(H_k + H_{n+k})}{k!(n+k)!}\left(\frac{x}{2}\right)^{2k+n}. \end{aligned}$$

The constant γ is called the **Euler-Máscheroni constant** or merely the **Euler gamma.**

$$\gamma \equiv \lim_{n\to\infty}(H_n - \ln n) \approx 0.577216.$$

Bessel's Equation of Order $p = 0$ (1 of 2)

We have already found

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}.$$

Assume a second solution linearly independent from $J_0(x)$ has the form

$$y_2(x) = (\ln x) J_0(x) + \sum_{k=1}^{\infty} c_k x^k,$$

where c_k for $k \in \mathbb{N}$ are constants.

It can be shown that

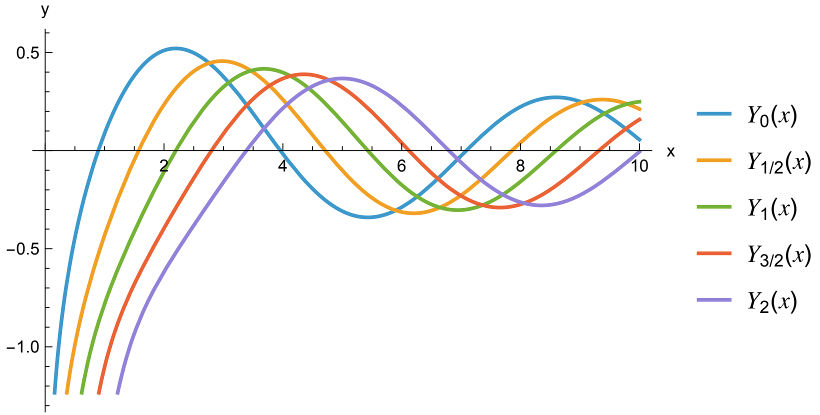
$$y_2(x) = (\ln x) J_0(x) - \sum_{k=1}^{\infty} \frac{(-1)^k H_k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}.$$

Bessel's Equation of Order $p = 0$ (2 of 2)

By convention the Bessel function of the second kind of order 0 is expressed as

$$Y_0(x) = \frac{2}{\pi}\left(\gamma + \ln\frac{x}{2}\right)J_0(x) - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k H_k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}.$$

Bessel Functions of the Second Kind



Derivative Relationships

Theorem

The Bessel functions of the first kind of order p satisfy the following derivative relationships:

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x),$$

$$\frac{d}{dx}\left[x^{p}J_p(x)\right] = x^{p}J_{p-1}(x).$$

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Corollary

The Bessel functions of the first kind obey the following recurrence relations.

$$xJ_p'(x) - pJ_p(x) = -xJ_{p+1}(x)$$
$$xJ_p'(x) + pJ_p(x) = xJ_{p-1}(x)$$
$$J_{p+1}(x) + J_{p-1}(x) = \frac{2p}{x}J_p(x)$$

Zeros of Bessel Functions

Lemma

The positive zeros of $J_0(x)$ form a strictly increasing sequence which tends to positive infinity.

Theorem

Let n be a nonnegative integer and suppose $\{\lambda_{n,k}\}_{k=1}^{\infty}$ are the positive zeros of $J_n(x)$ arranged in increasing order. Between each consecutive pair of zeros of $J_n(x)$ there exists a unique zero of $J_{n+1}(x)$.

Parametric Form of Bessel's Equation

The **parametric form of Bessel's Equation** is

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (\lambda^2 z^2 - p^2) y(z) = 0$$

The function $J_p(x)$ solves Bessel's equation of order p if and only if the function $J_p(\lambda z)$ solves the parametric form of Bessel's equation of order p.

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The function $J_p(x)$ solves Bessel's equation of order p if and only if the function $J_p(\lambda z)$ solves the parametric form of Bessel's equation of order p.

Theorem

Let $p \geq 0$ and let $\lambda_{p,k}$ be the kth zero of $J_p(x)$, then

$$\int_0^1 J_p(\lambda_{p,n} z) J_p(\lambda_{p,m} z) z \, dz = \frac{1}{2} \left(J_{p+1}(\lambda_{p,n}) \right)^2 \delta_{mn},$$

*for $m, n \in \mathbb{N}$, where δ_{mn} is the **Kronecker delta function**.*

Bessel-Fourier Series

If $f(x)$ is sufficiently smooth on $[0, 1]$, $f(x)$ can be expressed as

$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_{p,n} x),$$

where

$$a_n = \frac{2 \int_0^1 f(x) J_p(\lambda_{p,n} x) x \, dx}{\left(J_{p+1}(\lambda_{p,n})\right)^2}.$$

Bessel Functions of Half-Integer Order (1 of 3)

The **Bessel functions of half-integer order** are the functions $J_{n+\frac{1}{2}}(x)$ and $Y_{n+\frac{1}{2}}(x)$ where n is an integer.

Using the formula for $J_p(x)$ with $p = 1/2$,

$$
\begin{aligned}
J_{\frac{1}{2}}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1+1/2)} \left(\frac{x}{2}\right)^{2k+\frac{1}{2}} \\
&= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k! \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \\
&= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sqrt{\frac{2}{\pi x}} \sin x.
\end{aligned}
$$

Bessel Functions of Half-Integer Order (2 of 3)

Using the derivative realtionships between Bessel functions implies

$$\begin{aligned}
J_{3/2}(x) &= \frac{1}{2x} J_{\frac{1}{2}}(x) - J'_{\frac{1}{2}}(x) \\
&= \frac{1}{2x}\sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x + \frac{1}{2x}\sqrt{\frac{2}{\pi x}} \sin x \\
&= \sqrt{\frac{2}{\pi x}} \left(\frac{1}{x} \sin x - \cos x \right).
\end{aligned}$$

Bessel Functions of Half-Integer Order (3 of 3)

Theorem

For $n = 0, 1, 2, \ldots$

$$J_{n+\frac{1}{2}}(x) = (-1)^n x^{n+1} \sqrt{\frac{2}{\pi x}} \left(\frac{d}{x\, dx} \right)^n \left[\frac{\sin x}{x} \right].$$

Theorem

For $n = 0, 1, 2, \ldots$

$$J_{-n-\frac{1}{2}}(x) = x^{n+1} \sqrt{\frac{2}{\pi x}} \left(\frac{d}{x\, dx} \right)^n \left[\frac{\cos x}{x} \right].$$

Spherical Bessel's Equation

The **spherical Bessel's equation** of order n is

$$x^2 y'' + 2x\,y' + (x^2 - n(n+1))y = 0.$$

The solutions are the **spherical Bessel function of the first kind**

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$$

and the **spherical Bessel function of the second kind**

$$y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+\frac{1}{2}}(x).$$

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Corollary

For $n = 0, 1, 2, \ldots$,

$$\int_0^1 j_n(\lambda_{n+\frac{1}{2},k} x) j_n(\lambda_{n+\frac{1}{2},l} x) x^2\, dx = \frac{1}{2} \left(j_{n+1}(\lambda_{n+\frac{1}{2},k}) \right)^2 \delta_{kl}$$

where δ_{kl} *is the Kronecker delta function.*