Hermite Equation

Partial Differential Equations

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Fall 2025

Objectives

In this lesson we will:

- explore the Hermite ordinary differential equation,
- find two linearly independent solutions to Hermite's equation,
- define the Hermite polynomials, and
- determine some properties of the Hermite polynomials.

Hermite's Ordinary Differential Equation

$$y'' - 2xy' + \lambda y = 0 \text{ for } -\infty < x < \infty,$$

where λ is a constant, is known as the **Hermite equation**. This is an important equation in mathematical physics and is often encountered in applications involving the quantum harmonic oscillator.

Two linearly independent solutions to Hermite's differential equation are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n}(4k - 4 - \lambda)}{(2n)!} x^{2n}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n}(4k - 2 - \lambda)}{(2n+1)!} x^{2n+1}.$$

Hermite Polynomials

If $\lambda = 2m$ for some $m \in \mathbb{N}$ then one of the infinite series solutions terminates after a finite number of terms and can be written as a polynomial. In this case the ordinary differential equation is called the **Hermite equation of order** m.

If m is even (or equivalently if $\lambda = 4N$ for some $N \in \{0, 1, 2, \ldots\}$) then

$$y_1(x) = 1 + \sum_{n=1}^{N} \frac{\prod_{k=1}^{n}(4k - 4 - 4N)}{(2n)!} x^{2n}.$$

If m is odd (or equivalently if $\lambda = 4N - 2$ for some $N \in \mathbb{N}$) then

$$y_2(x) = x + \sum_{n=1}^{N} \frac{\prod_{k=1}^{n}(4k - 4N)}{(2n+1)!} x^{2n+1}.$$

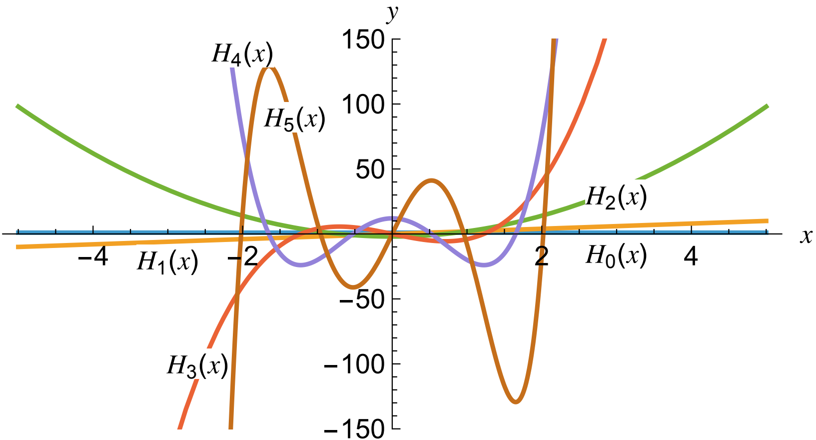
Examples

The **Hermite polynomial** denoted as $H_n(x)$ is defined to be the polynomial solution to the Hermite equation with $\lambda = 2n$ for which the coefficient of x^n is 2^n. The even- and odd-degree Hermite polynomials can be represented by the single formula:

$$H_n(x) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!(n-2k)!}(2x)^{n-2k}.$$

$$\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \\
H_4(x) &= 16x^4 - 48x^2 + 12 \\
H_5(x) &= 32x^5 - 160x^3 + 120x
\end{aligned}$$

Illustrations



Properties

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left[e^{-x^2}\right]$$

Theorem

The Hermite polynomials $H_n(x)$ are orthogonal on the interval $(-\infty, \infty)$ with respect to the weighting function e^{-x^2}. For $m,\ n \in \{0, 1, 2, \ldots\}$,

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2}\, dx = 2^n n! \sqrt{\pi}\,\delta_{mn},$$

where δ_{mn} is the Kronecker delta function.