Laguerre Equation

Partial Differential Equations

J Robert Buchanan

Department of Mathematics

Fall 2025

Objectives

In this lesson we will learn:

- to solve the Laguerre ordinary differential equation,
- to define the Laguerre polynomials,
- the properties of the Laguerre polynomials,
- to solve the associated Laguerre ordinary differential equation, and
- the associated Laguerre polynomials.

Laguerre Ordinary Differential Equation

$$xy'' + (1 - x)y' + \lambda y = 0 \text{ for } x > 0.$$

The solutions to the Laguerre equation find applications in quantum mechanics, numerical integration, and applied mathematics.

Using the method of Frobenius one solution to Laguerre's equation is

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{\prod_{k=0}^{n-1}(k - \lambda)}{(n!)^2} x^n.$$

Laguerre Polynomials

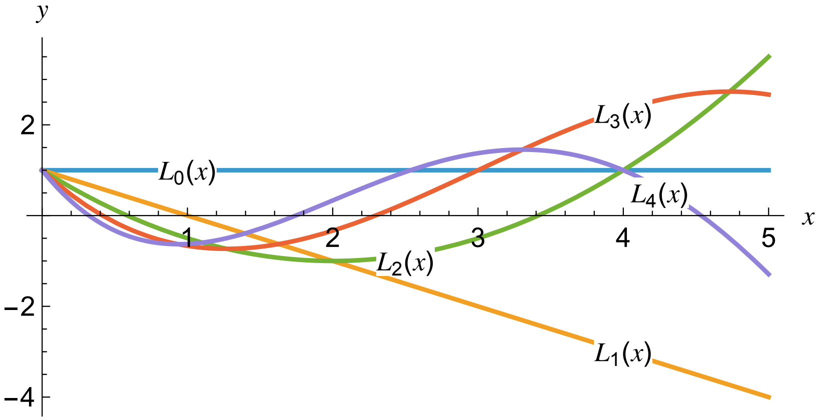
When $\lambda = m \in \{0, 1, 2, \dots\}$ then $y_1(x)$ contains powers of x less than or equal to m. The solution $y_1(x)$ is traditionally denoted as $L_m(x)$ and

$$\begin{aligned} L_m(x) &= 1 + \sum_{n=1}^{m} \frac{\prod_{k=0}^{n-1}(k-m)}{(n!)^2} x^n \\ &= 1 + \sum_{n=1}^{m} \frac{(-1)^n m(m-1)(m-2)\cdots(m-(n-1))}{(n!)^2} x^n \\ &= m! \sum_{n=0}^{m} \frac{(-1)^n}{(n!)^2 (m-n)!} x^n. \end{aligned}$$

Examples

$$
\begin{aligned}
L_0(x) &= 1 \\
L_1(x) &= 1 - x \\
L_2(x) &= 1 - 2x + \frac{1}{2}x^2 \\
L_3(x) &= 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3 \\
L_4(x) &= 1 - 4x + 3x^2 - \frac{2}{3}x^3 + \frac{1}{24}x^4
\end{aligned}
$$

Illustrations



Properties

Lemma

For $n = 0, 1, 2, \ldots$ the Rodrigues' form of the nth Laguerre polynomial is

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}[e^{-x}x^n].$$

Theorem

The Laguerre polynomials are orthogonal on $[0, \infty)$ with respect to the weighting function e^{-x}. For all nonnegative integers m and n, then

$$\int_0^\infty L_m(x)L_n(x)e^{-x}\,dx = \delta_{mn},$$

where δ_{mn} is the Kronecker delta function.

Generalized Fourier Series

If $f(x)$ is sufficiently smooth on the interval $[0, \infty)$ and if

$$\int_0^\infty (f(x))^2 \, e^{-x} \, dx < \infty,$$

then $f(x)$ may be represented as a generalized Fourier series (a Laguerre series) in terms of the Laguerre polynomials:

$$f(x) = \sum_{n=0}^{\infty} a_n L_n(x).$$

The coefficients a_n of the series may be found from the integral formula

$$a_n = \int_0^\infty f(x) L_n(x) e^{-x} \, dx$$

for $n = 0, 1, \ldots$.

Associated Laguerre Differential Equation

The **associated Laguerre differential equation** is

$$xy'' + (\alpha + 1 - x)y' + \lambda y = 0 \text{ for } x \geq 0,$$

where $\alpha > -1$ is a real number and λ is also a real number. The associated Laguerre equation often appears when solving the Schrödinger equation of quantum mechanics.

A general infinite series solution to the associated Laguerre equation is

$$y_1(x) = \frac{\Gamma(1+\lambda+\alpha)}{\Gamma(1+\lambda)\Gamma(1+\alpha)} \sum_{k=1}^{\infty} \frac{(-1)^k (\lambda - k + 1)_k}{k!\,(1+\alpha)_k} x^k.$$

Associated Laguerre Polynomials

When $\lambda = n \in \mathbb{N} \cup \{0\}$, the series coefficients simplify to

$$a_k = \frac{(-1)^k \Gamma(1+n+\alpha)}{k!(n-k)!\Gamma(1+k+\alpha)} \text{ for } k = 0, 1, \ldots, n,$$

and $a_k = 0$ for $k \geq n+1$. Thus one solution to the associated Laguerre equation is the polynomial

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(-1)^k \Gamma(1+n+\alpha) x^k}{k!(n-k)!\Gamma(1+k+\alpha)}.$$

It should be noted that when $\alpha = 0$ the associated Laguerre equation simplifies to the original Laguerre equation and

$$L_n^0(x) = L_n(x).$$

Examples

$$
\begin{aligned}
L_0^\alpha(x) &= 1 \\
L_1^\alpha(x) &= 1 + \alpha - x \\
L_2^\alpha(x) &= 1 + \tfrac{3}{2}\alpha + \tfrac{1}{2}\alpha^2 - (2+\alpha)x + \tfrac{1}{2}x^2 \\
L_3^\alpha(x) &= 1 + \tfrac{11}{6}\alpha + \alpha^2 + \tfrac{1}{6}\alpha^3 - \left(3 + \tfrac{5}{2}\alpha\right)x + \left(\tfrac{3}{2} + \tfrac{1}{2}\alpha\right)x^2 - \tfrac{1}{6}x^3
\end{aligned}
$$

Properties

Lemma

The associated Laguerre polynomial can be expressed as

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} [e^{-x} x^{n+\alpha}].$$

Theorem

The associated Laguerre polynomials $L_n^\alpha(x)$, are orthogonal on $[0, \infty)$ with respect to the weighting function $e^{-x} x^\alpha$. For all nonnegative integers m and n,

$$\int_0^\infty L_m^\alpha(x) L_n^\alpha(x) e^{-x} x^\alpha \, dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{mn},$$

where δ_{mn} is the Kronecker delta function.

Generalized Fourier Series

Suitably smooth functions can be represented as **generalized Laguerre series**,

$$f(x) = \sum_{n=0}^{\infty} a_n L_n^{\alpha}(x),$$

where

$$a_n = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^{\infty} f(x) L_n^{\alpha}(x) e^{-x} x^{\alpha} \, dx$$

is the n**th generalized Laguerre coefficient.**