Legendre's Equation

Partial Differential Equations

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Objectives

In this lesson we will learn:

- to solve Legendre's differential equation,
- properties of the Legendre polynomials,
- the relationships to the associated Legendre polynomials,
- the associated Legendre functions, and
- the orthogonality relationships.

Legendre's Differential Equation

When using the method of separation of variables to solve Laplace's equation in spherical coordinates, the following equation arises,

$$\frac{d^2F(\varphi)}{d\varphi^2} + \cot\varphi \frac{dF(\varphi)}{d\varphi} + \alpha(\alpha+1)F(\varphi) = 0$$

for $0 < \varphi < \pi$, where α is a constant.

Make the change of variable $x = \cos\varphi$ and define $y = f(x) = F(\arccos x)$. The equation above can be written in the form known as **Legendre's differential equation**,

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \text{ for } -1 < x < 1,$$

where α is a constant.

Solution to Legendre's Differential Equation

Assuming a power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ produces two linearly independent solutions,

$$y_1(x) = 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 + \sum_{m=3}^{\infty} \frac{\alpha\cdots(\alpha-2m+2)(\alpha+1)\cdots(\alpha+2m-1)}{(2m)!}x^{2m}$$

$$y_2(x) = x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5 + \sum_{m=3}^{\infty}(-1)^m \frac{(\alpha-1)\cdots(\alpha-2m+1)(\alpha+2)\cdots(\alpha+2m)}{(2m+1)!}x^{2m+1}.$$

If $\alpha \in \mathbb{N} \cup \{0\}$ then these functions are polynomials of degree α.

Leading Coefficient Convention

By convention the leading coefficient a_n of a polynomial solution of degree n to Legendre's differential equation is chosen to be

$$a_n = \frac{(2n)!}{2^n (n!)^2}.$$

In this situation a polynomial of degree n solving Legendre's differential equation and having the prescribed leading coefficient is called a **Legendre polynomial** and is denoted as $P_n(x)$. The general formula for $P_n(x)$ is

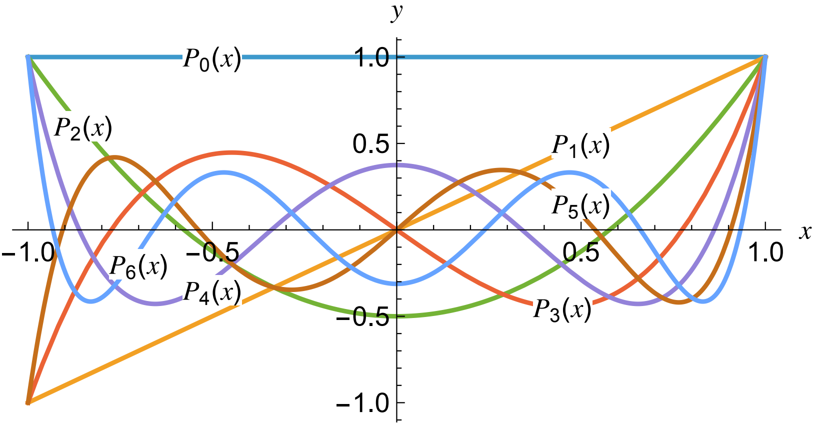
$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k},$$

where $\lfloor n/2 \rfloor$ denotes the greatest integer less than or equal to $n/2$.

Table of Legendre Polynomials

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x) \\
P_6(x) &= \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)
\end{aligned}
$$

Illustration



Properties of Legendre Polynomials

Theorem (Rodrigues' Form of Legendre Polynomials)

For $n = 0, 1, 2, \ldots$,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2 - 1)^n \right].$$

Lemma (Bonnet's Recurrence Relation)

For $n \in \mathbb{N}$, *if* $P_n(x)$ *is the nth Legendre polynomial*

$$(n+1)P_{n+1}(x) + nP_{n-1}(x) = (2n+1)xP_n(x).$$

Theorem

For $n, m \in \{0, 1, 2, \ldots\}$, *if* $P_m(x)$ *and* $P_n(x)$ *are the mth and nth Legendre polynomials respectively, then*

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1} \delta_{mn}$$

where δ_{mn} *is the Kronecker delta function.*

Associated Legendre Functions (1 of 3)

Theorem

Let $f(x) = P_n(x)$ then for $m \in \{0, 1, \dots, n\}$ the function $f^{(m)}(x)$ solves the ordinary differential equation,

$$(1 - x^2)y'' - 2x(m + 1)y' + (n(n + 1) - m(m + 1))y = 0.$$

Associated Legendre Functions (1 of 3)

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Define the function $g(x) = (1 - x^2)^{m/2} f^{(m)}(x)$. Function $g(x)$ solves the ordinary differential equation,

$$(1 - x^2)g''(x) - 2xg'(x) + \left(n(n+1) - \frac{m^2}{1 - x^2}\right) g(x) = 0$$

which is called the **associated Legendre equation**. The solutions are referred to as the **associated Legendre functions**, and are denoted as $P_n^m(x) = (-1)^m (1 - x^2)^{m/2} P_n^{(m)}(x)$ for $n = 0, 1, \ldots$ and $m = 0, 1, \ldots, n$ where $P_n(x)$ is the nth Legendre polynomial.

Associated Legendre Functions (2 of 3)

	$m=0$	$m=1$	$m=2$	$m=3$
$n=0$	1			
$n=1$	x	$-\sqrt{1-x^2}$		
$n=2$	$\frac{1}{2}(-1+3x^2)$	$-3x\sqrt{1-x^2}$	$-3x^2+3$	
$n=3$	$\frac{1}{2}(-3x+5x^3)$	$-\frac{3}{2}(-1+5x^2)\sqrt{1-x^2}$	$-15x(-1+x^2)$	$-15(1-x^2)^{3/2}$

Associated Legendre Functions (3 of 3)

$$P_n^m(x) = (-1)^m (1 - x^2)^{m/2} P_n^{(m)}(x)$$

The integer parameter m is called the order of the associated Legendre function. If $0 < m \leq n$ the associated Legendre function for $-m < 0$ is defined as a scalar multiple of the associated Legendre function of order $m > 0$,

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x) \text{ for } 0 < m \leq n.$$

Lemma

If $n + m$ is even then $P_n^m(x)$ is an even function and if $n + m$ is odd then $P_n^m(x)$ is an odd function.

Properties

Theorem

Suppose m and n are nonnegative integers such that $m \neq n$ and suppose k is an integer such that $|k| \leq \min\{m, n\}$. Then

$$\int_{-1}^{1} P_m^k(x) P_n^k(x)\, dx = 0.$$

Theorem

If $|k| \leq n$, then

$$\int_{-1}^{1} \left(P_n^k(x)\right)^2\, dx = \frac{2}{2n+1} \frac{(n+k)!}{(n-k)!}.$$

Orthogonality

If $f(x)$ is sufficiently smooth on $[-1, 1]$, the associated Legendre function expansion of $f(x)$ of order k is

$$f(x) = \sum_{n=k}^{\infty} a_n P_n^k(x),$$

where

$$a_n = \frac{2n+1}{2} \frac{(n-k)!}{(n+k)!} \int_{-1}^{1} f(x) P_n^k(x)\, dx$$

for $n = k, k+1, k+2, \ldots$. If $k = 0$ this reduces to the Legendre function expansion.