Spherical Harmonics

Partial Differential Equations

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Objectives

In this lesson we will learn:

- the eigenfunctions and eigenvalues of Laplace's equation in spherical coordinates,
- the properties of the spherical harmonic functions.

Laplace's Equation in Spherical Coordinates

Consider Laplace's equation in spherical coordinates (ρ, φ, θ), (note that coordinate φ is the zenith angle between the positive z-axis and the line formed by connecting a point in three-dimensional space to the origin. As such, $0 \leq \varphi \leq \pi$. The coordinate θ is the azimuthal coordinate as in the polar coordinate system. This is conventional notation for mathematicians though physicists and engineers often reverse the interpretations of φ and θ).

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\left(\frac{\partial^2 u}{\partial \varphi^2} + \cot\varphi \frac{\partial u}{\partial \varphi} + \csc^2\varphi \frac{\partial^2 u}{\partial \theta^2}\right) = 0.$$

Assume a product solution of the form $u(\rho, \varphi, \theta) = R(\rho)\Phi(\varphi)\Theta(\theta)$.

$$\begin{aligned} 0 = {} & R''(\rho)\Phi(\varphi)\Theta(\theta) + \frac{2}{\rho}R'(\rho)\Phi(\varphi)\Theta(\theta) + \frac{1}{\rho^2}R(\rho)\Phi''(\varphi)\Theta(\theta) \\ & + \frac{1}{\rho^2}\left(\cot\varphi R(\rho)\Phi'(\varphi)\Theta(\theta) + \csc^2\varphi R(\rho)\Phi(\varphi)\Theta''(\theta)\right) \end{aligned}$$

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = \sin^2\varphi\left(\rho^2\frac{R''(\rho)}{R(\rho)} + 2\rho\frac{R'(\rho)}{R(\rho)} + \frac{\Phi''(\varphi)}{\Phi(\varphi)} + \cot\varphi\frac{\Phi'(\varphi)}{\Phi(\varphi)}\right)$$

Solutions (1 of 3)

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = \sin^2\varphi \left(\rho^2 \frac{R''(\rho)}{R(\rho)} + 2\rho \frac{R'(\rho)}{R(\rho)} + \frac{\Phi''(\varphi)}{\Phi(\varphi)} + \cot\varphi \frac{\Phi'(\varphi)}{\Phi(\varphi)}\right)$$

The right-hand side of the equation depends on ρ and φ and the left-hand side depends on θ, thus both sides equal a constant, denoted as σ. Consequently one of the ordinary differential equations implied by the separation of variables procedure is

$$\Theta''(\theta) + \sigma\Theta(\theta) = 0.$$

The solution to Laplace's equation should be 2π-periodic in variable θ, thus the needed constant is $\sigma = m^2$ where $m = 0, 1, 2, \ldots$. For convenience express the general solution as a linear combination of the complex exponentials $\Theta(\theta) = c_1 e^{i\,m\,\theta} + c_2 e^{-i\,m\,\theta}$ where $i = \sqrt{-1}$.

Solutions (2 of 3)

$$\sin^2\varphi\left(\rho^2\frac{R''(\rho)}{R(\rho)}+2\rho\frac{R'(\rho)}{R(\rho)}+\frac{\Phi''(\varphi)}{\Phi(\varphi)}+\cot\varphi\frac{\Phi'(\varphi)}{\Phi(\varphi)}\right)=m^2,$$

the variables ρ and φ can be separated as follows

$$\rho^2\frac{R''(\rho)}{R(\rho)}+2\rho\frac{R'(\rho)}{R(\rho)}=-\frac{\Phi''(\varphi)}{\Phi(\varphi)}-\cot\varphi\frac{\Phi'(\varphi)}{\Phi(\varphi)}+m^2\csc^2\varphi.$$

The right-hand side of the equation is a function of φ only and the left-hand -side is a function of ρ only, thus both sides are constant, denoted as λ.

Solutions (3 of 3)

Consider the equation:

$$-\frac{\Phi''(\varphi)}{\Phi(\varphi)} - \cot\varphi \frac{\Phi'(\varphi)}{\Phi(\varphi)} + m^2 \csc^2\varphi = \lambda$$
$$\Phi''(\varphi) + \cot\varphi \Phi'(\varphi) + (\lambda - m^2 \csc^2\varphi)\Phi(\varphi) = 0,$$

where λ is a constant and make the change of variable $x = \cos\varphi$. The equation above can be rewritten as

$$(1 - x^2)\frac{d^2\Phi}{dx^2} - 2x\frac{d\Phi}{dx} + \left(\lambda - \frac{m^2}{1 - x^2}\right)\Phi = 0.$$

If $\lambda = n(n+1)$ where $n = 0, 1, 2, \ldots$, this becomes the associated Legendre differential equation.
Define

$$f(\varphi, \theta) = \Phi(\varphi)\Theta(\theta) = e^{i\,m\,\theta} P_n^m(\cos\varphi)$$

with $n = 0, 1, 2, \ldots$ and $-n \leq m \leq n$.

Spherical Harmonic Functions

	$m=0$	$m=1$	$m=2$	$m=3$
$n=0$	$\frac{1}{2\sqrt{\pi}}$			
$n=1$	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\varphi$	$-\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\,\theta}\sin\varphi$		
$n=2$	$\frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\varphi-1)$	$-\frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{i\,\theta}\cos\varphi\sin\varphi$	$\frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{2i\,\theta}\sin^2\varphi$	
$n=3$	$\frac{1}{4}\sqrt{\frac{7}{\pi}}(5\cos^3\varphi-3\cos\varphi)$	$-\frac{1}{8}\sqrt{\frac{21}{\pi}}e^{i\,\theta}(5\cos^2\varphi-1)\sin\varphi$	$\frac{1}{4}\sqrt{\frac{105}{2\pi}}e^{2i\,\theta}\cos\varphi\sin^2\varphi$	$-\frac{1}{8}\sqrt{\frac{35}{\pi}}e^{3i\,\theta}\sin^3\varphi$

Properties

Theorem

Let $n = 0, 1, 2, \ldots$ *and let* m *be an integer with* $-n \leq m \leq n$*. Likewise let* $\hat{n} = 0, 1, 2, \ldots$ *and let* $\hat{m}$ *be an integer with* $-\hat{n} \leq \hat{m} \leq \hat{n}$ *then*

$$\int_{-\pi}^{\pi} \int_{0}^{\pi} e^{i\,m\,\theta} P_n^m(\cos\varphi) e^{-i\,\hat{m}\,\theta} P_{\hat{n}}^{\hat{m}}(\cos\varphi) \sin\varphi \, d\varphi \, d\theta = 0$$

if $n \neq \hat{n}$ *or* $m \neq \hat{m}$*. On the other hand if* $n = \hat{n}$ *and* $m = \hat{m}$*, then*

$$\int_{-\pi}^{\pi} \int_{0}^{\pi} \left(P_n^m(\cos\varphi)\right)^2 \sin\varphi \, d\varphi \, d\theta = \frac{4\pi}{2n+1} \frac{(n+m)!}{(n-m)!}.$$

Spherical Harmonic Functions

For integers $n = 0, 1, 2, \ldots$ and $-n \leq m \leq n$ the normalized function $f(\varphi, \theta)$ is denoted as $Y_n^m(\varphi, \theta)$ where

$$Y_n^m(\varphi, \theta) = \sqrt{\frac{2n+1}{4\pi} \, \frac{(n-m)!}{(n+m)!}} e^{i\,m\,\theta} P_n^m(\cos\varphi)$$

These functions are called the **spherical harmonic functions.** There are $2n+1$ orthonormal functions corresponding to the eigenvalue $n(n+1)$.