Vibration of a Circular Drumhead

Partial Differential Equations

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Objectives

In this lesson we will:

- model the vibration of a circular drumhead,
- use separation of variables to solve the initial boundary value problem, and
- use a generalized Fourier series to express the formal solution describing the displacement of the drumhead.

Assumptions and Model

A flexible circular membrane of radius $r_0 > 0$ is centered at the origin and has its circular edge fixed so that it cannot be displaced from the xy-plane. Suppose the initial displacement of the membrane for $x^2 + y^2 < r_0^2$ is described by the function $f(x, y)$ and the initial velocity of the membrane is 0. The displacement u of the membrane from the xy-plane can be described by the initial boundary value problem:

$$
\begin{aligned}
u_{tt} &= c^2(u_{xx} + u_{yy}) \text{ for } x^2 + y^2 < r_0^2 \text{ and } t > 0 \\
u(x, y, t) &= 0 \text{ for } x^2 + y^2 = r_0^2 \text{ and } t > 0 \\
u(x, y, 0) &= f(x, y) \text{ for } x^2 + y^2 < r_0^2 \\
u_t(x, y, 0) &= 0 \text{ for } x^2 + y^2 < r_0^2.
\end{aligned}
$$

The constant $c > 0$ is the wave speed of vibrations in the membrane and is directly proportional to the square root of the tension present in the membrane and inversely proportional to the square root of the density (mass per unit area) of the membrane.

Polar Coordinates

The circular geometry of the membrane suggests that the solution may be more easily found using the polar coordinate system. Re-writing the problem in polar coordinates produces

$$u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right) \text{ for } r < r_0,\ \theta \in \mathbb{R},\ t > 0$$

$$u(r_0, \theta, t) = 0 \text{ for } \theta \in \mathbb{R} \text{ and } t > 0$$

$$u(r, \theta, 0) = f(r, \theta) \text{ for } r < r_0 \text{ and } \theta \in \mathbb{R}$$

$$u_t(r, \theta, 0) = 0 \text{ for } r < r_0 \text{ and } \theta \in \mathbb{R}.$$

Separation of Variables (1 of 2)

Assume a product solution of the form $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$.

$$R(r)\Theta(\theta)T''(t) = c^2\left(R''(r)\Theta(\theta)T(t) + \frac{R'(r)\Theta(\theta)T(t)}{r} + \frac{R(r)\Theta''(\theta)T(t)}{r^2}\right)$$

$$\frac{T''(t)}{c^2T(t)} = \frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)}.$$

Since the left-hand side of the equation depends on t alone while the right-hand side depends on (r,θ), then both sides must be constant.

Suppose the constant $-\lambda^2 \leq 0$.

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = -\lambda^2$$

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \lambda^2r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)}.$$

Separation of Variables (2 of 2)

$$r^2 \frac{R''(r)}{R(r)} + r \frac{R'(r)}{R(r)} + \lambda^2 r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)}$$

The right-hand side of the equation is a function of θ only and the solution to the initial boundary value problem should be 2π-periodic in θ, thus both sides of the equation are equal to the constant m^2 where $m = 0, 1, 2, \ldots$.

$$\Theta(\theta) \equiv \Theta_m(\theta) = c_1 \cos(m\theta) + c_2 \sin(m\theta),$$

where c_1 and c_2 are constant coefficients.

Separation of Variables (2 of 2)

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The ordinary differential equation implied by the left-hand side of the separated variables equation can be written as

$$r^2 R''(r) + rR'(r) + (\lambda^2 r^2 - m^2)R(r) = 0.$$

This is the parametric form of Bessel's equation of order m. Thus the general solution is $R(r) = c_3 J_m(\lambda r) + c_4 Y_m(\lambda r)$.

Bounding the Solution

- The solution should be bounded as $r \to 0^+$, thus $R_m(r) = J_m(\lambda r)$.
- The boundary condition $u(r_0, \theta, t) = 0$ implies $J_m(\lambda r_0) = 0$ for each m, and hence $\lambda \equiv \lambda_{m,n}/r_0$ where $\lambda_{m,n}$ is the nth zero of the Bessel function of the first kind of order m.

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The ordinary differential equation implied by the time dependent portion of the PDE is

$$T''(t) + \left(\frac{c\lambda_{m,n}}{r_0}\right)^2 T(t) = 0.$$

As a result, the time dependent portion of the solution can be written as

$$T(t) = c_5 \cos\left(\frac{c\lambda_{m,n}t}{r_0}\right) + c_6 \sin\left(\frac{c\lambda_{m,n}t}{r_0}\right).$$

Product Solution

$$u_{m,n}(r,\theta,t) = J_m\left(\frac{\lambda_{m,n} r}{r_0}\right)(c_1\cos(m\theta) + c_2\sin(m\theta))$$
$$\left(c_5\cos\left(\frac{c\lambda_{m,n} t}{r_0}\right) + c_6\sin\left(\frac{c\lambda_{m,n} t}{r_0}\right)\right)$$

for $m = 0, 1, 2, \ldots$ and $n \in \mathbb{N}$.

Product Solution

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for $m = 0, 1, 2, \ldots$ and $n \in \mathbb{N}$.

By the Principle of Superposition,

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[J_m\left(\frac{\lambda_{m,n} r}{r_0}\right)\left(A_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)\right)\cos\left(\frac{c\lambda_{m,n} t}{r_0}\right)\right]$$

where $A_{m,n}$ and $B_{m,n}$ are coefficients which will be determined from the initial displacement of the membrane.

Determining the Coefficients (1 of 2)

The coefficients can be found by setting $t = 0$.

$$f(r,\theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m\left(\frac{\lambda_{m,n} r}{r_0}\right)\left[A_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)\right]$$

Multiply both sides by $\pi^{-1}\cos(k\theta)$ and integrate over $[-\pi,\pi]$. The only nonzero term in the summation over index m will be the term for which $m = k$, thus

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\cos(k\theta)\,d\theta = \sum_{n=1}^{\infty} A_{k,n} J_k\left(\frac{\lambda_{k,n} r}{r_0}\right).$$

Similarly if both sides are multiplied by $\pi^{-1}\sin(k\theta)$ and integrated over $[-\pi,\pi]$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\sin(k\theta)\,d\theta = \sum_{n=1}^{\infty} B_{k,n} J_k\left(\frac{\lambda_{k,n} r}{r_0}\right).$$

Determining the Coefficients (2 of 2)

Coefficients can now be found individually using Bessel series coefficient formulas. Make the substitution $z = r/r_0$. In the special case where $k = 0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(r_0 z, \theta)\, d\theta = \sum_{n=1}^{\infty} A_{0,n} J_0(\lambda_{0,n} z).$$

Using the Bessel series coefficient formula,

$$A_{0,n} = \frac{1}{\pi (J_1(\lambda_{0,n}))^2}\int_0^1 z\, J_0(\lambda_{0,n} z)\left(\int_{-\pi}^{\pi} f(r_0 z, \theta)\, d\theta\right)\, dz.$$

For $k \in \mathbb{N}$,

$$A_{k,n} = \frac{2\int_0^1 z\, J_k(\lambda_{k,n} z)\left(\int_{-\pi}^{\pi} f(r_0 z, \theta)\cos(k\theta)\, d\theta\right)\, dz}{\pi (J_{k+1}(\lambda_{k,n}))^2}$$

$$B_{k,n} = \frac{2\int_0^1 z\, J_k(\lambda_{k,n} z)\left(\int_{-\pi}^{\pi} f(r_0 z, \theta)\sin(k\theta)\, d\theta\right)\, dz}{\pi (J_{k+1}(\lambda_{k,n}))^2}.$$

Fundamental Frequency

The motion of the vibrating membrane can be thought of as the superposition of a number of modes of vibration. The (m, n)th mode of vibration is given by the expression,

$$J_m\left(\frac{\lambda_{m,n} r}{r_0}\right)\left(A_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)\right)\cos\left(\frac{c\lambda_{m,n} t}{r_0}\right).$$

The frequency associated with this mode is

$$f_{m,n} = \frac{c\lambda_{m,n}}{2\pi r_0}$$

and therefore the fundamental frequency corresponding to $m = 0$ and $n = 1$ is $f_{0,1} = c\lambda_{0,1}/(2\pi r_0)$.

Example

Find the displacement from equilibrium for the circular drumhead described in the following initial boundary value problem.

$$
\begin{aligned}
u_{tt} &= \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right) \text{ for } r < 1,\ \theta \in \mathbb{R},\ t > 0 \\
u(1, \theta, t) &= 0 \text{ for } \theta \in \mathbb{R} \text{ and } t > 0 \\
u(r, \theta, 0) &= r(1 - r^2) \sin\theta \text{ for } r < 1 \text{ and } \theta \in \mathbb{R} \\
u_t(r, \theta, 0) &= 0 \text{ for } r < 1 \text{ and } \theta \in \mathbb{R}.
\end{aligned}
$$

Solution (1 of 3)

Since $f(r, \theta)$ is an odd function in θ, the coefficients $A_{k,n} = 0$ for $k = 0, 1, 2, \ldots$ and $n \in \mathbb{N}$. The orthogonality of the trigonometric functions on $[-\pi, \pi]$ implies $B_{k,n} = 0$ for $k = 2, 3, \ldots$ and $n \in \mathbb{N}$. Thus the infinite series solution simplifies considerably to the form,

$$u(r, \theta, t) = \sin(\theta) \sum_{n=1}^{\infty} B_{1,n} J_1(\lambda_{1,n} r) \cos(\lambda_{1,n} t).$$

Solution (2 of 3)

The coefficients $B_{1,n}$ for $n \in \mathbb{N}$ can be found by integration

$$B_{1,n} = \frac{2}{\pi (J_2(\lambda_{1,n}))^2} \int_0^1 z\, J_1(\lambda_{1,n} z) \left(\int_{-\pi}^{\pi} z(1 - z^2) \sin^2(\theta)\, d\theta \right)\, dz$$
$$= \frac{2}{(J_2(\lambda_{1,n}))^2} \int_0^1 (z^2 - z^4)\, J_1(\lambda_{1,n} z)\, dz = \frac{16}{\lambda_{1,n}^3 J_2(\lambda_{1,n})}.$$

Solution (2 of 3)

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The formal solution can then be expressed as

$$u(r, \theta, t) = 16 \sin(\theta) \sum_{n=1}^{\infty} \frac{J_1(\lambda_{1,n} r)}{\lambda_{1,n}^3 J_2(\lambda_{1,n})} \cos(\lambda_{1,n} t).$$

Solution (3 of 3)