Heat Equation in a Sphere

Partial Differential Equations

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Objectives

In this lesson we will:

- use spherical coordinates to model the diffusion of heat energy throughout a sphere,
- use separation of variables to determine a product solution to the hear equation in a sphere, and
- use the orthogonality properties of eigenfunctions to determine the series coefficients for a solution to the initial boundary value problem.

Mathematical Model

Consider a solid sphere of radius a constructed of a homogeneous material. The surface of the sphere is kept at temperature 0 and at time $t = 0$ the initial distribution of temperature within the sphere is described by the function $f(\rho, \varphi, \theta)$ where (ρ, φ, θ) are the spherical coordinates. If $u(\rho, \varphi, \theta, t)$ denotes the temperature in the sphere at time t, then an initial boundary value problem which models this situation can be written as,

$$
\begin{aligned}
u_t &= \kappa \Delta u \text{ for } 0 \leq \rho < a,\ 0 < \varphi < \pi,\ -\pi < \theta < \pi,\ t > 0 \\
u(a, \varphi, \theta, t) &= 0 \text{ for } 0 < \varphi < \pi,\ -\pi < \theta < \pi,\ t > 0 \\
u(\rho, \varphi, \theta, 0) &= f(\rho, \varphi, \theta) \text{ for } 0 \leq \rho < a,\ 0 < \varphi < \pi,\ -\pi < \theta < \pi.
\end{aligned}
$$

Separation of Variables (1 of 2)

Assume a product solution of the form $u(\rho, \varphi, \theta, t) = T(t)R(\rho)F(\varphi, \theta)$ exists for the partial differential equation. The variables can be separated as follows:

$$
\begin{aligned}
T'(t)R(\rho)F(\varphi,\theta) &= \kappa T(t)R''(\rho)F(\varphi,\theta) + \frac{2\kappa}{\rho} T(t)R'(\rho)F(\varphi,\theta) \\
&\quad + \frac{\kappa}{\rho^2}\left(TRF_{\varphi\varphi} + \cot\varphi\, TRF_{\varphi} + \csc^2\varphi\, TRF_{\theta\theta}\right) \\
\frac{T'(t)}{\kappa T(t)} &= \frac{R''(\rho)}{R(\rho)} + \frac{2}{\rho}\frac{R'(\rho)}{R(\rho)} + \frac{1}{\rho^2 F}\left(F_{\varphi\varphi} + \cot\varphi F_{\varphi} + \csc^2\varphi F_{\theta\theta}\right).
\end{aligned}
$$

The left-hand side of the equation depends only on t while the right-hand side depends only on the spatial variables (ρ, φ, θ). Hence both sides are equal to a constant denoted as $-\lambda$.

Separation of Variables (2 of 2)

Working on the right-hand side of the last equation, the variables can be further separated as

$$-\lambda = \frac{R''(\rho)}{R(\rho)} + \frac{2}{\rho}\frac{R'(\rho)}{R(\rho)} + \frac{1}{\rho^2 F(\varphi,\theta)}\left(F_{\varphi\varphi} + \cot\varphi F_{\varphi} + \csc^2\varphi F_{\theta\theta}\right)$$

$$\frac{\Delta F(\varphi,\theta)}{F(\varphi,\theta)} = -\lambda\rho^2 - \frac{\rho^2 R''(\rho)}{R(\rho)} - \frac{2\rho R'(\rho)}{R(\rho)}.$$

The numerator on left-hand side of the last equation involves the Laplacian operator applied to a function of (φ,θ). The right-hand side depends only on ρ. Again both sides must equal a constant.

The differential equation induced by the left-hand side of the last equation has the spherical harmonic functions $Y_n^m(\varphi,\theta)$ as solutions and eigenvalues $n(n+1)$ for $n = 0, 1, 2, \ldots$.

Radial Factor of Solution

The implied ordinary differential equation involving the independent variable ρ is

$$\rho^2 R''(\rho) + 2\rho R'(\rho) + (\lambda\rho^2 - n(n+1))R(\rho) = 0.$$

Thinking ahead to the time-dependent portion of the product solution, it is expected for λ to be positive. Based on prior exposure to the heat equation with homogeneous Dirichlet boundary conditions and the physical intuition that, in the absence of sources of heat energy within the sphere, the asymptotic temperature distribution within the sphere should be zero. Thus $T(t)$ should decay asymptotically to zero. To clarify the notation, then replace λ with ν^2 where $\nu > 0$.

Consider the differential equation now written as

$$\rho^2 R''(\rho) + 2\rho R'(\rho) + (\nu^2\rho^2 - n(n+1))R(\rho) = 0.$$

Finding the Radial Solution

Make the change of variable $x = \nu\rho$, then

$$x^2 \frac{d^2 R}{dx^2} + 2x \frac{dR}{dx} + (x^2 - n(n+1))R(x) = 0$$

which is the spherical Bessel's equation. The general solution to the spherical Bessel's equation is

$$R_n(x) = A_n j_n(x) + B_n y_n(x).$$

However, since $y_n(x)$ is unbounded as $x \to 0^+$, the coefficient B_n must be chosen to be 0.

Hence $R_n(\rho) = A_n j_n(\nu\rho)$.

Satisfying the Boundary Conditions

The homogeneous Dirichlet boundary condition requires $R_n(a) = 0$ and thus $\nu a = \lambda_{n+1/2,q}$, the qth zero of the Bessel function of the first kind of order $n + 1/2$.

Finally the radial portion of the product solution may be expressed as,

$$R_{n,m}(\rho) = j_n(\lambda_{n+1/2,q}\rho/a).$$

With $\lambda = \nu^2 = \lambda^2_{n+1/2,q}$, the time-dependent ordinary differential equation can be expressed as

$$\frac{T'(t)}{\kappa T(t)} = -\left(\frac{\lambda_{n+1/2,q}}{a}\right)^2 \implies T_{n,q}(t) = e^{-\kappa\lambda^2_{n+1/2,q}t/a^2}.$$

Product Solution

Multiplying the factors of the product solution produces

$$u_{n,m,q}(\rho, \varphi, \theta, t) = j_n\left(\frac{\lambda_{n+1/2,q}\rho}{a}\right) Y_n^m(\varphi, \theta) e^{-\kappa \lambda_{n+1/2,q}^2 t/a^2}.$$

By the Principle of Superposition, a formal solution to the heat equation on the sphere can be expressed as a triple infinite series,

$$u(\rho, \varphi, \theta, t) = \sum_{n=0}^{\infty} \sum_{q=1}^{\infty} \sum_{m=-n}^{n} A_{n,m,q} j_n\left(\frac{\lambda_{n+1/2,q}\rho}{a}\right) Y_n^m(\varphi, \theta) e^{-\kappa \lambda_{n+1/2,q}^2 t/a^2}.$$

The coefficients $A_{n,m,q}$ are determined by the initial temperature distribution within the sphere.

Initial Conditions

When $t = 0$,

$$f(\rho, \varphi, \theta) = \sum_{n=0}^{\infty} \sum_{q=1}^{\infty} \sum_{m=-n}^{n} A_{n,m,q} j_n \left(\frac{\lambda_{n+1/2,q}\rho}{a} \right) Y_n^m(\varphi, \theta).$$

Multiply by $\overline{U_{n,m,q}(\rho, \varphi, \theta)}\rho^2 \sin\varphi$ and integrate over the sphere of radius a.

$$A_{n,m,q} = \frac{2}{a^3 \left(j_{n+1}(\lambda_{n+1/2,q}) \right)^2} \int_0^a \int_0^\pi \int_{-\pi}^{\pi} f(\rho, \varphi, \theta) \overline{U_{n,m,q}(\rho, \varphi, \theta)} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$$

Example

Find the temperature distribution within a sphere described by the following initial boundary value problem.

$$
\begin{aligned}
u_t &= \kappa \Delta u \text{ for } 0 \leq \rho < a,\ 0 < \varphi < \pi,\ -\pi < \theta < \pi,\ t > 0 \\
u(a, \varphi, \theta, t) &= 0 \text{ for } 0 < \varphi < \pi,\ -\pi < \theta < \pi,\ t > 0 \\
u(\rho, \varphi, \theta, 0) &= j_2\left(\frac{\lambda_{5/2,1}\rho}{a}\right)\left(P_2(\cos\varphi) + \sin(2\theta) P_2^2(\cos\varphi)\right) \\
&\quad \text{for } 0 \leq \rho < a,\ 0 < \varphi < \pi,\ -\pi < \theta < \pi.
\end{aligned}
$$

Solution

Quite a great deal of work is saved if the initial condition is written in the equivalent form

$$f(\rho,\varphi,\theta) = j_2\left(\frac{\lambda_{5/2,1}\rho}{a}\right)\left(\sqrt{\frac{4\pi}{5}}\,Y_2^0(\varphi,\theta) + 2i\sqrt{\frac{6\pi}{5}}\,(Y_2^{-2}(\varphi,\theta) - Y_2^2(\varphi,\theta))\right).$$

Once the initial condition is expressed in terms of eigenfunctions, the solution can be expressed as

$$\begin{aligned}
u(\rho,\varphi,\theta,t) &= \sqrt{\frac{4\pi}{5}}\,Y_2^0(\varphi,\theta)A_{2,0,1}j_2\left(\frac{\lambda_{5/2,1}\rho}{a}\right)e^{-\kappa\lambda_{5/2,1}^2 t/a^2} \\
&\quad + 2i\sqrt{\frac{6\pi}{5}}\,Y_2^{-2}(\varphi,\theta)A_{2,-2,1}j_2\left(\frac{\lambda_{5/2,1}\rho}{a}\right)e^{-\kappa\lambda_{5/2,1}^2 t/a^2} \\
&\quad - 2i\sqrt{\frac{6\pi}{5}}\,Y_2^{2}(\varphi,\theta)A_{2,2,1}j_2\left(\frac{\lambda_{5/2,1}\rho}{a}\right)e^{-\kappa\lambda_{5/2,1}^2 t/a^2} \\
&= e^{-\kappa\lambda_{5/2,1}^2 t/a^2} f(\rho,\varphi,\theta).
\end{aligned}$$