Modeling a Swinging Chain

Partial Differential Equations

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Objectives

In this lesson we will:

- model the motion of a swinging chain, and
- solve the partial differential equation modeling a swinging chain.

Developing a Mathematical Model

- Consider a chain or string with one end fixed at the origin of a rectangular coordinate system, with the positive x-axis oriented in the downward direction.
- The length of the chain is L and the linear density of the chain is $\rho(x)$.
- The chain is subject to gravitational acceleration g (again in the downward, positive x-direction).
- If the chain is unperturbed by other forces, it will remain still, occupying the interval $[0, L]$ along the x-axis.
- If the chain is moved by a small amount from this equilibrium position, let $u(x, t)$ denote the displacement of the chain from the x-axis at position x and time t.

Tension in the Chain

The tension at position x is

$$T(x) = g \int_x^L \rho(s)\, ds.$$

Proceeding as in the development of the model for a vibrating string,

$$\rho(x)u_{tt} = -g\rho(x)u_x + T(x)u_{xx} + f(x,t)$$

where $f(x,t)$ represents any external forces on the chain.

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If $f(x,t) = 0$ the initial, boundary value problem for the swinging chain is

$$\begin{aligned}
\rho(x)u_{tt} &= (T(x)u_x)_x \\
u(0,t) &= 0 \text{ for } t > 0 \\
u(x,0) &= f(x) \text{ and } u_t(x,0) = g(x) \text{ for } 0 < x < L.
\end{aligned}$$

Constant Linear Density

Suppose $T(x) = g\rho(L - x)$ where ρ is the constant linear density.

This simplifies the partial differential equation to

$$u_{tt} = -gu_x + g(L - x)u_{xx}.$$

Assuming a product solution of the form $u(x, t) = X(x)T(t)$,

$$\frac{T''(t)}{gT(t)} = -\frac{X'(x)}{X(x)} + (L - x)\frac{X''(x)}{X(x)} = \gamma,$$

where γ is a constant. Thus the variables can be separated.

The boundary value problem for $X(x)$ can be expressed as

$$\begin{aligned} (L - x)X''(x) - X'(x) - \gamma X(x) &= 0 \text{ for } 0 < x < L \\ X(0) &= 0. \end{aligned}$$

Change of Variable

Remark: any physically meaningful solution to the initial boundary value problem of the swinging chain must be bounded. This implies that $\gamma < 0$.

Let $L - x = \alpha \xi^{\beta}$ where α and β are constants. Making the change of variable results in

$$\xi^{2-\beta} \frac{d^2 X}{d\xi^2} + \xi^{1-\beta} \frac{dX}{d\xi} - \alpha \beta^2 \gamma X = 0.$$

If $\beta = 2$, $\alpha = -1/(4\gamma)$, and both sides of the ODE are multiplied by ξ^2 then Bessel's equation of order zero results.

$$\xi^2 \frac{d^2 X}{d\xi^2} + \xi \frac{dX}{d\xi} + \xi^2 X = 0.$$

Consequently the general solution can be expressed as $X(\xi) = c_1 J_0(\xi) + c_2 Y_0(\xi)$.

Bounding the Solution

- The solution must be bounded as $x \to L^-$ which is equivalent to $\xi \to 0^+$ and implies $c_2 = 0$.
- In terms of the original independent variable $X(x) = c_1 J_0(2\sqrt{-\gamma(L-x)})$.
- Choose $\gamma = -\lambda^2$ with $\lambda > 0$ and impose the boundary condition at $u(0,t) = 0$ for all $t > 0$ implying $J_0(2\lambda\sqrt{L}) = 0$ which in turn implies that $2\lambda\sqrt{L}$ is a zero of the Bessel function of the first kind of order zero.
- Solve the time-dependent portion of separated equation to yield $T(t) = c_3 \cos\left(\sqrt{g}\lambda t\right) + c_4 \sin\left(\sqrt{g}\lambda t\right)$.
- Hence the product solutions indexed by $\lambda > 0$ take the form

$$u_\lambda(x,t) = \left[A_\lambda \cos\left(\sqrt{g}\lambda t\right) + B_\lambda \sin\left(\sqrt{g}\lambda t\right)\right] J_0(2\lambda\sqrt{L-x}),$$

where A_λ and B_λ are constants.

Principle of Superposition

The formal solution to the swinging chain can be written as the generalized Fourier series:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{\lambda_{0,n}}{2}\sqrt{\frac{g}{L}}\,t \right) + B_n \sin\left(\frac{\lambda_{0,n}}{2}\sqrt{\frac{g}{L}}\,t \right) \right] J_0\left(\lambda_{0,n}\sqrt{1-\frac{x}{L}} \right).$$

Example

Find the formal solution to the initial boundary value problem of the perturbed chain released from rest.

$$\begin{aligned} u_{tt} &= -gu_x + g(L - x)u_{xx} \\ u(0, t) &= 0 \text{ for } t > 0 \\ u(x, 0) &= x \text{ and } u_t(x, 0) = 0 \text{ for } 0 < x < L. \end{aligned}$$

Solution (1 of 4)

$$u(x,t)=\sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{\lambda_{0,n}}{2}\sqrt{\frac{g}{L}}t\right)+B_n\sin\left(\frac{\lambda_{0,n}}{2}\sqrt{\frac{g}{L}}t\right)\right]J_0\left(\lambda_{0,n}\sqrt{1-\frac{x}{L}}\right).$$

Since the initial velocity of the chain is zero then $B_n=0$ for $n\in\mathbb{N}$. Setting $t=0$,

$$u(x,0)=x=\sum_{n=1}^{\infty}A_nJ_0\left(\lambda_{0,n}\sqrt{1-\frac{x}{L}}\right).$$

Recall the change of variable.

$$\xi=\sqrt{1-\frac{x}{L}}\iff x=L(1-\xi^2)$$

The initial displacement of the chain can be expressed as

$$L(1-\xi^2)=\sum_{n=1}^{\infty}A_nJ_0\left(\lambda_{0,n}\xi\right).$$

Solution (2 of 4)

Using the orthogonality of the Bessel functions,

$$
\begin{aligned}
A_n &= \frac{2\int_0^1 L(1-\xi^2)J_0(\lambda_{0,n}\xi)\xi\,d\xi}{(J_1(\lambda_{0,n}))^2} \\
&= \frac{2L}{(J_1(\lambda_{0,n}))^2}\left(\frac{1}{\lambda_{0,n}^2}\int_0^{\lambda_{0,n}} uJ_0(u)\,du - \frac{1}{\lambda_{0,n}^4}\int_0^{\lambda_{0,n}} u^2(uJ_0(u))\,du\right) \\
&= \frac{2L}{(J_1(\lambda_{0,n}))^2}\left(\frac{2}{\lambda_{0,n}^4}\int_0^{\lambda_{0,n}} u^2J_1(u)\,du\right) \\
&= \frac{4L}{\lambda_{0,n}^4(J_1(\lambda_{0,n}))^2}\left[u^2J_2(u)\right]_{u=0}^{u=\lambda_{0,n}} \\
&= \frac{4LJ_2(\lambda_{0,n})}{\lambda_{0,n}^2(J_1(\lambda_{0,n}))^2} \\
&= \frac{8L}{\lambda_{0,n}^3J_1(\lambda_{0,n})}.
\end{aligned}
$$

Solution (3 of 4)

We can numerically approximate as many of the zeros of the Bessel function of the first kind of order zero as necessary.

The formal solution takes the form,

$$u(x,t) = 8L \sum_{n=1}^{\infty} \frac{1}{\lambda_{0,n}^3 J_1(\lambda_{0,n})} J_0\left(\lambda_{0,n}\sqrt{1-\frac{x}{L}}\right).$$

Solution (4 of 4)