Duhamel's Principle

Partial Differential Equations

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Objectives

In this lesson we will:

- introduce Duhamel's principle, and
- show how Duhamel's principle can be used to solve the nonhomogeneous heat equation with homogeneous boundary and initial conditions.

Initial Boundary Value Problem

$$
\begin{aligned}
u_t &= \kappa\, u_{xx} + g(x,t) \text{ for } 0 < x < L \text{ and } t > 0 \\
u(0,t) &= 0 \text{ and } u(L,t) = 0 \text{ for } t > 0 \\
u(x,0) &= 0 \text{ for } 0 < x < L.
\end{aligned}
$$

Remark: the function $g(x,t)$ represents the rate of temperature change in the medium caused by an internal heat source.

Motivation of Duhamel's Principle

- Suppose the interval $[0, T]$ is partitioned into subintervals $\{[s_{i-1}, s_i]\}_{i=1}^{n}$ where
$$0 = s_0 \leq s_1 \leq \cdots \leq s_{i-1} \leq s_i \leq \cdots \leq s_n = T.$$
- For $i = 1, \ldots, n$ define $\Delta s_i = s_i - s_{i-1}$.
- Define the functions $\mathbb{1}_i(t)$ as
$$\mathbb{1}_i(t) = \begin{cases} 1 & \text{if } s_{i-1} \leq t < s_i, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the initial boundary value problem

$$\begin{aligned} u_t &= \kappa\, u_{xx} + \mathbb{1}_i(t) g(x,t) \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= 0 \text{ and } u(L,t) = 0 \text{ for } t > 0 \\ u(x,0) &= 0 \text{ for } 0 < x < L. \end{aligned}$$

Remark: the nonhomogeneity in the PDE is only present during the interval $s_{i-1} \leq t \leq s_i$.

Motivation of Duhamel's Principle

$$\begin{aligned} u_t &= \kappa\, u_{xx} + \mathbb{1}_i(t) g(x,t) \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= 0 \text{ and } u(L,t) = 0 \text{ for } t > 0 \\ u(x,0) &= 0 \text{ for } 0 < x < L. \end{aligned}$$

- For $i = 1, \ldots, n$ let $u_i(x,t)$ be the solution to the corresponding initial boundary value problem.
- For t in the interval $[0, T]$, $\sum_{i=1}^{n} \mathbb{1}_i(t) g(x,t) = g(x,t)$ and thus by the Principle of Superposition, $\sum_{i=1}^{n} u_i(x,t) = u(x,t)$ where $u(x,t)$ solves the original initial boundary value problem for $0 \leq t \leq T$.

Motivation of Duhamel's Principle

$$\begin{aligned} u_t &= \kappa\, u_{xx} + \mathbb{1}_i(t) g(x,t) \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= 0 \text{ and } u(L,t) = 0 \text{ for } t > 0 \\ u(x,0) &= 0 \text{ for } 0 < x < L. \end{aligned}$$

- Consider the three intervals, $[0, s_{i-1})$, $[s_{i-1}, s_i)$ and $[s_i, T]$.
- Since the boundary and initial conditions are zero $u_i(x,t) = 0$ for $0 \le t < s_{i-1}$ and $0 \le x \le L$.
- At $t = s_i$ the temperature distribution in the medium is approximately $g(x, s_{i-1})\Delta s_i$.
- For $t \ge s_i$ the heat source is turned off and a new initial boundary value problem can be considered.
- For the new problem, there is a nonzero, initial temperature distribution approximated as $g(x, s_{i-1})\Delta s_i$.

New Initial Boundary Value Problem

$$
\begin{aligned}
u_t &= \kappa\, u_{xx} \text{ for } 0 < x < L \text{ and } t > s_i \\
u(0,t) = 0 &\text{ and } u(L,t) = 0 \text{ for } t > s_i \\
u(x,s_i) &= g(x,s_{i-1})\Delta s_i \text{ for } 0 < x < L.
\end{aligned}
$$

For $t \geq s_i$ the contribution to the temperature distribution in the rod from the switching on of the heat source during the time interval $[s_{i-1}, s_i)$ is approximately $v(x,t)\Delta s_i$ where $v(x,t)$ is the solution to the following initial value problem:

$$
\begin{aligned}
v_t &= \kappa\, v_{xx} \text{ for } 0 < x < L \text{ and } t > s_i \\
v(0,t) = 0 &\text{ and } v(L,t) = 0 \text{ for } t > s_i \\
v(x,s_i) &= g(x,s_{i-1}) \text{ for } 0 < x < L.
\end{aligned}
$$

The dependence of v on the time at which the heat source is turned on, is expressed as $v(x,t;s)$.

New Initial Boundary Value Problem

$$\begin{aligned} v_t &= \kappa\, v_{xx} \text{ for } 0 < x < L \text{ and } t > s_i \\ v(0,t;s_{i-1}) &= v(L,t;s_{i-1}) = 0 \text{ for } t > s_i \\ v(x,s_i;s_{i-1}) &= g(x,s_{i-1}) \text{ for } 0 < x < L. \end{aligned}$$

- $u_i(x,t) = v(x,t;s_{i-1})\Delta s_i$ where $v(x,t;s_{i-1})$ solves the IBVP above.
- At $t = T$ the solution u to the original IBVP is

$$u(x,T) \approx \sum_{i=1}^{n} v(x,T;s_{i-1})\Delta s_i.$$

- As the norm of the partition approaches zero,

$$u(x,T) = \lim_{n\to\infty} \sum_{i=1}^{n} v(x,T;s_{i-1})\Delta s_i = \int_0^T v(x,T;s)\,ds.$$

Summary

Since T is arbitrary, the variable can be replaced by t and $v(x, t; s)$ is a solution to the following initial boundary value problem:

$$\begin{aligned} v_t &= \kappa\, v_{xx} \text{ for } 0 < x < L \text{ and } t > s \\ v(0, t; s) &= v(L, t; s) = 0 \text{ for } t > s \\ v(x, s; s) &= g(x, s) \text{ for } 0 < x < L. \end{aligned}$$

Change of Variable

Define $\hat{v}(x, t - s; s) = v(x, t; s)$, then $v_t = \hat{v}_t$ and $v_{xx} = \hat{v}_{xx}$ and hence $\hat{v}(x, t; s)$ solves the initial boundary value problem:

$$\begin{aligned}
\hat{v}_t &= \kappa\, \hat{v}_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\
\hat{v}(0, t; s) &= \hat{v}(L, t; s) = 0 \text{ for } t > 0 \\
\hat{v}(x, 0; s) &= g(x, s) \text{ for } 0 < x < L.
\end{aligned}$$

The solution $u(x, t)$ becomes

$$u(x, t) = \int_0^t v(x, t; s)\, ds = \int_0^t \hat{v}(x, t - s; s)\, ds.$$