Nonhomogeneous Wave Equation

Partial Differential Equations

J Robert Buchanan

Department of Mathematics

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Objectives

In this lesson we will:

- extend Duhamel's principle to the wave equation, and
- solve a nonhomogeneous wave equation using this method.

Unbounded String

Consider a vibrating string initially at rest in its equilibrium position,

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + g(x,t) \text{ for } -\infty < x < \infty \text{ and } t > 0 \\ u(x,0) &= u_t(x,0) = 0 \text{ for } -\infty < x < \infty. \end{aligned}$$

The nonhomogeneous term $g(x,t)$ can be interpreted as the acceleration of the string at position x at time t and thus is sometimes referred to as the **applied acceleration**.

Motivation of Duhamel's Principle

- Let $\{s_i\}_{i=0}^{n}$ be a partition of the interval $[0, T]$ such that

$$0 = s_0 \leq s_1 \leq \cdots \leq s_{i-1} \leq s_i \leq \cdots \leq s_n = T.$$

- For $i = 1, \ldots, n$ define $\Delta s_i = s_i - s_{i-1}$ and define the support functions $\mathbb{1}_i(t)$ as before.
- Suppose at time $t = s_{i-1}$ the applied acceleration is turned on and it is turned off at $t = s_i$.

The initial boundary value problem for the infinitely long string with external acceleration applied only during $[s_{i-1}, s_i)$ is

$$u_{tt} = c^2 u_{xx} + \mathbb{1}_i(t) g(x,t) \text{ for } -\infty < x < \infty \text{ and } t > 0$$
$$u(x,0) = u_t(x,0) = 0 \text{ for } -\infty < x < \infty.$$

If $u_i(x,t)$ is the solution to the corresponding initial value problem for $i = 1, \ldots, n$, then $\sum_{i=1}^{n} u_i(x,t) = u(x,t)$, the solution to the initial value problem for $0 \leq t \leq T$.

Motivation of Duhamel's Principle

- The solution $u_i(x, t) = 0$ for $0 \leq t < s_{i-1}$.
- If Δs_i is small then during $[s_{i-1}, s_i)$ the external force accelerates the string to a velocity of approximately $g(x, s_{i-1})\Delta s_i$.
- The external force displaces the string by an approximate amount $g(x, s_{i-1})(\Delta s_i)^2/2$ which can be ignored if Δs_i is small.

Thus for $t \geq s_i$ the contribution to the displacement of the string $u_i(x, t)$ can be approximated by the solution to the following initial value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > s_i \\ u(x, s_i) &= 0 \text{ for } -\infty < x < \infty \\ u_t(x, s_i) &= g(x, s_{i-1})\Delta s_i \text{ for } -\infty < x < \infty. \end{aligned}$$

Motivation for Duhamel's Principle

For $t > s_i$ the displacement $u_i(x,t) \approx v(x,t)\Delta s_i$, where $v(x,t)$ solves the initial value problem:

$$\begin{aligned} v_{tt} &= c^2 v_{xx} \text{ for } -\infty < x < \infty \text{ and } t > s_i \\ v(x, s_i; s_{i-1}) &= 0 \text{ for } -\infty < x < \infty \\ v_t(x, s_i; s_{i-1}) &= g(x, s_{i-1}) \text{ for } -\infty < x < \infty. \end{aligned}$$

To indicate the dependence of $v(x,t)$ on s, the solution will be denoted as $v(x, s_i; s_{i-1})$. Using the Riemann sum argument as in the case of the nonhomogeneous heat equation yields

$$u(x, T) = \int_0^T v(x, T; s)\, ds.$$

Motivation for Duhamel's Principle

T is arbitrary it can be replaced by any $t > 0$. As the norm of the partition approaches zero, $v(x,t;s)$ solves the following initial value problem:

$$\begin{aligned} v_{tt} &= c^2 v_{xx} \text{ for } -\infty < x < \infty \text{ and } t > s \\ v(x,s;s) &= 0 \text{ for } -\infty < x < \infty \\ v_t(x,s;s) &= g(x,s) \text{ for } -\infty < x < \infty. \end{aligned}$$

Define $\hat{v}(x,t-s;s) = v(x,t;s)$, where $\hat{v}$ solves the initial value problem:

$$\begin{aligned} \hat{v}_{tt} &= c^2 \hat{v}_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\ \hat{v}(x,0;s) &= 0 \text{ for } -\infty < x < \infty \\ \hat{v}_t(x,0;s) &= g(x,s) \text{ for } -\infty < x < \infty. \end{aligned}$$

Main Result

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + g(x,t) \text{ for } -\infty < x < \infty \text{ and } t > 0 \\ u(x,0) &= u_t(x,0) = 0 \text{ for } -\infty < x < \infty. \end{aligned}$$

Theorem

Suppose $g(x,t) \in \mathcal{C}^1(\mathbb{R} \times (0,\infty))$, then

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} g(r,s)\, dr\, ds$$

solves the initial value problem.

Example

Use Duhamel's principle to find the solution to the following initial value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + e^{-(x-t)} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\ u(x,0) &= \sin x \text{ for } -\infty < x < \infty \\ u_t(x,0) &= \cos x \text{ for } -\infty < x < \infty. \end{aligned}$$

Solution (1 of 3)

Using d'Alembert's approach,

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\ u(x,0) &= \sin x \text{ for } -\infty < x < \infty \\ u_t(x,0) &= \cos x \text{ for } -\infty < x < \infty, \end{aligned}$$

is solved by

$$\begin{aligned} u_1(x,t) &= \frac{1}{2}\left(\sin(x+ct) + \sin(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos s \, ds \\ &= \frac{1}{2}\left(\sin(x+ct) + \sin(x-ct)\right) + \frac{1}{2c}\left(\sin(x+ct) - \sin(x-ct)\right). \end{aligned}$$

Solution (2 of 3)

According to the previous theorem, the solution to the nonhomogeneous initial value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + e^{-(x-t)} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\ u(x,0) &= 0 \text{ for } -\infty < x < \infty \\ u_t(x,0) &= 0 \text{ for } -\infty < x < \infty, \end{aligned}$$

can be written as the double integral,

$$\begin{aligned} u_2(x,t) &= \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{-(r-s)}\,dr\,ds \\ &= \frac{1}{2c}\int_0^t \left(e^{-(x-ct)} e^{(1-c)s} - e^{-(x+ct)} e^{(c+1)s} \right) ds \\ &= \frac{1}{2c}\left(e^{-(x-ct)} \frac{e^{(1-c)t}-1}{1-c} - e^{-(x+ct)} \frac{e^{(1+c)t}-1}{1+c} \right). \end{aligned}$$

Solution (3 of 3)

$$u(x,t) = \frac{1}{2}\left(\sin(x+ct) + \sin(x-ct)\right) + \frac{1}{2c}\left(\sin(x+ct) - \sin(x-ct)\right)$$
$$+ \frac{1}{2c}\left(e^{-(x-ct)}\frac{e^{(1-c)t}-1}{1-c} - e^{-(x+ct)}\frac{e^{(1+c)t}-1}{1+c}\right)$$

Bounded Finite Length String

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} + F(x,t) \text{ for } 0 < x < L \text{ and } t > 0 \\
u(0,t) &= \phi(t) \text{ and } u(L,t) = \psi(t) \text{ for } t > 0 \\
u(x,0) &= f(x) \text{ and } u_t(x,0) = g(x) \text{ for } 0 < x < L.
\end{aligned}
$$

The reference function defined as

$$
u_1(x,t) = (\psi(t) - \phi(t))\,\frac{x}{L} + \phi(t)
$$

agrees with the nonhomogeneous boundary conditions at $x = 0$ and $x = L$.

Decomposing the Solution

The solution to the nonhomogeneous equation is $u(x,t) = u_1(x,t) + v(x,t)$ where $v(x,t)$ satisfies the following initial boundary value problem:

$$
\begin{aligned}
v_{tt} &= c^2 v_{xx} + F(x,t) \\
&\qquad - (\psi''(t) - \phi''(t))\,\frac{x}{L} - \phi''(t) \text{ for } 0 < x < L,\ t > 0 \\
v(0,t) &= v(L,t) = 0 \text{ for } t > 0 \\
v(x,0) &= f(x) - (\psi(0) - \phi(0))\,\frac{x}{L} - \phi(0) \text{ for } 0 < x < L \\
v_t(x,0) &= g(x) - (\psi'(0) - \phi'(0))\,\frac{x}{L} - \phi'(0) \text{ for } 0 < x < L.
\end{aligned}
$$

Decomposing the Solution

Let $v(x,t) = v_1(x,t) + v_2(x,t)$ where

$$
\begin{aligned}
(v_1)_{tt} &= c^2 (v_1)_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\
v_1(0,t) &= v_1(L,t) = 0 \text{ for } t > 0 \\
v_1(x,0) &= f(x) - (\psi(0) - \phi(0))\,\frac{x}{L} - \phi(0) \text{ for } 0 < x < L \\
(v_1)_t(x,0) &= g(x) - (\psi'(0) - \phi'(0))\,\frac{x}{L} - \phi'(0) \text{ for } 0 < x < L,
\end{aligned}
$$

and

$$
\begin{aligned}
(v_2)_{tt} &= c^2 (v_2)_{xx} + F(x,t) \\
&\qquad - (\psi''(t) - \phi''(t))\,\frac{x}{L} - \phi''(t) \text{ for } x \in (0,L),\ t > 0 \\
v_2(0,t) &= v_2(L,t) = 0 \text{ for } t > 0 \\
v_2(x,0) &= (v_2)_t(x,0) = 0 \text{ for } 0 < x < L.
\end{aligned}
$$

Decomposing the Solution

- Solution $v_1(x,t)$ can be found using a Fourier series or d'Alembert's approach.
- Solution $v_2(x,t)$ can be found using Duhamel's principle where

$$v_2(x,t) = \int_0^t \hat{v}(x, t-s; s)\, ds$$

Function $\hat{v}(x,t;s)$ solves the auxiliary problem:

$$\begin{aligned}
\hat{v}_{tt} &= c^2 \hat{v}_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\
\hat{v}(0,t;s) &= \hat{v}(L,t;s) = 0 \text{ for } t > 0 \\
\hat{v}(x,0;s) &= 0 \text{ for } 0 < x < L \\
\hat{v}_t(x,0;s) &= F(x,s) - (\psi''(s) - \phi''(s))\,\frac{x}{L} - \phi''(s) \text{ for } 0 < x < L.
\end{aligned}$$

Example

Use Duhamel's principle to find the solution to the initial boundary value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= 0 \text{ and } u(L,t) = \sin(\omega t) \text{ for } t > 0 \\ u(x,0) &= u_t(x,0) = 0 \text{ for } 0 < x < L. \end{aligned}$$

Solution (1 of 5)

The reference function is $u_1(x,t) = (x/L)\sin(\omega t)$ and thus if $u(x,t) = v(x,t) + u_1(x,t)$, then $v(x,t)$ solves the following initial boundary value problem:

$$v_{tt} = c^2 v_{xx} + \frac{\omega^2 x}{L}\sin(\omega t) \text{ for } 0 < x < L \text{ and } t > 0$$

$$v(0,t) = v(L,t) = 0 \text{ for } t > 0$$

$$v(x,0) = 0 \text{ and } v_t(x,0) = -\frac{\omega x}{L} \text{ for } 0 < x < L.$$

Solution (2 of 5)

The solution to the homogeneous portion of the initial boundary value problem

$$\begin{aligned} v_{tt} &= c^2 v_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ v(0,t) &= v(L,t) = 0 \text{ for } t > 0 \\ v(x,0) &= 0 \text{ and } v_t(x,0) = -\frac{\omega x}{L} \text{ for } 0 < x < L. \end{aligned}$$

was found to be

$$v_1(x,t) = \frac{2\omega L}{c\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin \frac{n\pi c\, t}{L} \sin \frac{n\pi\, x}{L}.$$

Solution (3 of 5)

The auxiliary problem is

$$\begin{aligned}
\hat{v}_{tt} &= c^2 \hat{v}_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\
\hat{v}(0,t;s) &= \hat{v}(L,t;s) = 0 \text{ for } t > 0 \\
\hat{v}(x,0;s) &= 0 \text{ for } 0 < x < L \\
\hat{v}_t(x,0;s) &= \frac{\omega^2 x}{L} \sin(\omega s) \text{ for } 0 < x < L.
\end{aligned}$$

The solution can be written as the Fourier series,

$$\hat{v}(x,t;s) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi ct}{L} \sin \frac{n\pi x}{L}$$

where

$$a_n = \frac{2(-1)^{n+1} L\omega^2}{n^2\pi^2 c} \sin(\omega s).$$

Solution (4 of 5)

According to Duhamel's principle the solution $v_2(x,t) = \int_0^t \hat{v}(x, t-s; s)\, ds$. Assuming the Fourier series can be integrated term by term, then

$$\int_0^t a_n \sin \frac{n\pi c(t-s)}{L} \sin \frac{n\pi x}{L}\, ds = b_n \sin \frac{n\pi x}{L} \int_0^t \sin(\omega s) \sin \frac{n\pi c(t-s)}{L}\, ds$$

where for $n \in \mathbb{N}$, $b_n = 2(-1)^{n+1} L\omega^2/(n^2\pi^2 c)$.

$$\begin{aligned}
&\int_0^t \sin(\omega s) \sin \frac{n\pi c(t-s)}{L}\, ds \\
&= \begin{cases} \dfrac{L^2\omega \sin(cn\pi\, t/L) - cLn\pi \sin(\omega t)}{(L\omega)^2 - (cn\pi)^2} & \text{if } n \neq \omega L/(c\pi), \\ \dfrac{\sin(\omega t)}{2\omega} - \dfrac{t}{2}\cos(\omega t) & \text{if } n = \omega L/(c\pi). \end{cases}
\end{aligned}$$

Solution (5 of 5)

If there exists no $N \in \mathbb{N}$ such that $N = \omega L/(c\pi)$ then

$$v_2(x,t) = \sum_{n=1}^{\infty} b_n \frac{L^2\omega \sin(cn\pi\, t/L) - cLn\pi \sin(\omega t)}{(L\omega)^2 - (cn\pi)^2} \sin \frac{n\pi x}{L}.$$

If there exists $N \in \mathbb{N}$ for which $N = \omega L/(c\pi)$ then

$$\begin{aligned} v_2(x,t) &= b_N \left(\frac{\sin(\omega t)}{2\omega} - \frac{t}{2}\cos(\omega t) \right) \sin \frac{\omega x}{c} \\ &\quad + \sum_{n=1, n\neq N}^{\infty} b_n \frac{L^2\omega \sin(cn\pi\, t/L) - cLn\pi \sin(\omega t)}{(L\omega)^2 - (cn\pi)^2} \sin \frac{n\pi x}{L}. \end{aligned}$$

The solution to the original IBVP is

$$\begin{aligned} u(x,t) &= v_2(x,t) + \frac{x}{L}\sin(\omega t) \\ &\quad + \frac{2\omega L}{c\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin \frac{n\pi c\, t}{L} \sin \frac{n\pi\, x}{L}. \end{aligned}$$