Fourier Sine and Cosine Transform

Partial Differential Equations

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Objectives

In this lesson we will:

- define the Fourier cosine and sine transforms,
- define the inverse Fourier cosine and sine transforms, and
- use the Fourier cosine and sine transforms to solve partial differential equations on the half line and half plane.

Fourier Cosine Transform and Inverse

Definition

If $f(x)$ is defined on $(0, \infty)$, then the **Fourier cosine transform** and the **inverse Fourier cosine transform** are defined respectively as follows.

$$\mathcal{F}_c[f](\omega) = \hat{f}^c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x)\, dx$$

$$\mathcal{F}_c^{-1}[F](x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\omega) \cos(\omega x)\, d\omega$$

Both transforms require that the improper integrals converge.

Fourier Sine Transform

Definition

If $f(x)$ is defined on $(0,\infty)$, then the **Fourier sine transform** and the **inverse Fourier sine transform** are defined as

$$\mathcal{F}_s[f](\omega) = \hat{f}^s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(\omega x)\,dx$$

$$\mathcal{F}_s^{-1}[F](x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\omega)\sin(\omega x)\,d\omega$$

Remark: the transformed function is even-extended for the Fourier cosine transform and odd-extended for the Fourier sine transform.

Example

Calculate the Fourier sine transform of $f(x) = e^{-x}$.

Solution

$$\begin{aligned}
\mathcal{F}_s[e^{-x}](\omega) &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin(\omega x)\, dx \\
&= \sqrt{\frac{2}{\pi}} \left[\frac{-e^{-x}(\omega \cos(\omega x) - \sin(\omega x))}{1+\omega^2} \right]_{x=0}^{x\to\infty} \\
&= \sqrt{\frac{2}{\pi}} \frac{\omega}{1+\omega^2}.
\end{aligned}$$

Properties

$$
\begin{aligned}
\mathcal{F}_c[f'](\omega) &= \omega \mathcal{F}_s[f](\omega) - \sqrt{\frac{2}{\pi}} f(0) \\
\mathcal{F}_s[f'](\omega) &= -\omega \mathcal{F}_c[f](\omega) \\
\mathcal{F}_c[f''](\omega) &= -\omega^2 \mathcal{F}_c[f](\omega) - \sqrt{\frac{2}{\pi}} f'(0) \\
\mathcal{F}_s[f''](\omega) &= -\omega^2 \mathcal{F}_s[f](\omega) + \sqrt{\frac{2}{\pi}} \omega f(0)
\end{aligned}
$$

Example

Consider the heat equation on the half-line.

$$
\begin{aligned}
u_t &= \kappa u_{xx} \text{ for } 0 < x < \infty \text{ and } t > 0 \\
u(x,0) &= x^2 e^{-x} \text{ for } 0 < x < \infty \\
u(0,t) &= 0 \text{ for } t > 0
\end{aligned}
$$

Use a transform method to find the solution to the initial boundary value problem.

Solution (1 of 3)

Since the boundary value $u(0, t)$ is known, the Fourier sine transform can be applied to this IBVP.

$$\begin{aligned}
\mathcal{F}_s[u_t(x,t)](\omega) &= \mathcal{F}_s[\kappa u_{xx}(x,t)](\omega) \\
\hat{u}^s_t(\omega,t) &= \kappa \left(-\omega^2 \hat{u}^s(\omega,t) + \sqrt{\frac{2}{\pi}}\omega u(0,t) \right) \\
&= -\kappa\omega^2 \hat{u}^s(\omega,t).
\end{aligned}$$

Solution (2 of 3)

$$\hat{u}^s_t(\omega, t) = -\kappa\omega^2 \hat{u}^s(\omega, t)$$

The general solution to this first-order, linear ordinary differential equation in variable t with unknown $\hat{u}^s$ is $\hat{u}^s(\omega, t) = A(\omega)e^{-\kappa\omega^2 t}$. The expression

$$A(\omega) = \mathcal{F}_s[u(x, 0)](\omega) = \mathcal{F}_s[x^2 e^{-x}](\omega) = 2\sqrt{\frac{2}{\pi}}\left(\frac{3-\omega^2}{(1+\omega^2)^3}\right).$$

Solution (3 of 3)

$$\hat{u}_s(\omega, t) = 2\sqrt{\frac{2}{\pi}} \left(\frac{3 - \omega^2}{(1 + \omega^2)^3} \right) e^{-\kappa\omega^2 t}$$

$$u(x, t) = \mathcal{F}_s^{-1} \left[2\sqrt{\frac{2}{\pi}} \left(\frac{3 - \omega^2}{(1 + \omega^2)^3} \right) e^{-\kappa\omega^2 t} \right] (x)$$

$$= \frac{4}{\pi} \int_0^\infty \frac{(3 - \omega^2)}{(1 + \omega^2)^3} e^{-\kappa\omega^2 t} \sin(\omega x) \, d\omega$$

Example

Consider the following boundary value problem for Laplace's equation on the positive quadrant of the Cartesian plane.

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0 \text{ for } 0 < x < \infty \text{ and } 0 < y < \infty \\
u(0, y) &= e^{-y} \sin y \text{ for } 0 < y < \infty \\
u_y(x, 0) &= e^{-x^2} \sin x \text{ for } 0 < x < \infty \\
\lim_{x \to \infty} u(x, y) &= 0 \text{ for } 0 < y < \infty \\
\lim_{y \to \infty} u(x, y) &= 0 \text{ for } 0 < x < \infty
\end{aligned}
$$

Solution (1 of 6)

Let $u(x,y) = u_1(x,y) + u_2(x,y)$ where $u_1(x,y)$ satisfies the following boundary value problem.

$$
\begin{aligned}
(u_1)_{xx} + (u_1)_{yy} &= 0 \text{ for } 0 < x < \infty \text{ and } 0 < y < \infty \\
u_1(0,y) &= e^{-y} \sin y \text{ for } 0 < y < \infty \\
(u_1)_y(x,0) &= 0 \text{ for } 0 < x < \infty \\
\lim_{x\to\infty} u_1(x,y) &= 0 \text{ for } 0 < y < \infty \\
\lim_{y\to\infty} u_1(x,y) &= 0 \text{ for } 0 < x < \infty
\end{aligned}
$$

Solution (2 of 6)

Calculate the Fourier cosine transform of the partial differential equation with respect to the variable y.

$$\mathcal{F}_c[(u_1)_{xx} + (u_1)_{yy}](\omega) = \mathcal{F}_c[0](\omega)$$

$$\frac{d^2}{dx^2}[\hat{u}_1^c(x,\omega)] - \omega^2 \hat{u}_1^c(x,\omega) = 0$$

The general solution to this ordinary differential equation is

$$\hat{u}_1^c(x,\omega) = A(\omega)e^{-\omega x} + B(\omega)e^{\omega x},$$

where A and B are currently functions of ω to be determined by the boundary conditions.

Solution (3 of 6)

$\hat{u}_1^c(x,\omega) \to 0$ as $x \to \infty$ since $\lim_{x\to\infty} u_1(x,y) = 0$. This implies $B(\omega) = 0$ for all $\omega > 0$. Hence $\hat{u}_1^c(x,\omega) = A(\omega)e^{-\omega x}$ for all $x \geq 0$.

In particular when $x = 0$,

$$\hat{u}_1(0,\omega) = A(\omega) = \mathcal{F}_c[e^{-y}\sin y](\omega) = \frac{4-2\omega^2}{(4+\omega^4)\sqrt{2\pi}}$$

which implies

$$\hat{u}_1^c(x,\omega) = \frac{4-2\omega^2}{(4+\omega^4)\sqrt{2\pi}}e^{-\omega x}.$$

Taking the inverse cosine transform yields

$$u_1(x,y) = \mathcal{F}_c^{-1}[\hat{u}_1^c(x,\omega)](y) = \frac{1}{\pi}\int_0^\infty \frac{4-2\omega^2}{4+\omega^4}e^{-\omega x}\cos(\omega y)\,d\omega.$$

Solution (4 of 6)

Function $u_2(x, y)$ must satisfy the following boundary value problem.

$$
\begin{aligned}
(u_2)_{xx} + (u_2)_{yy} &= 0 \text{ for } 0 < x < \infty \text{ and } 0 < y < \infty \\
u_2(0, y) &= 0 \text{ for } 0 < y < \infty \\
(u_2)_y(x, 0) &= e^{-x^2} \sin x \text{ for } 0 < x < \infty \\
\lim_{x\to\infty} u_2(x, y) &= 0 \text{ for } 0 < y < \infty \\
\lim_{y\to\infty} u_2(x, y) &= 0 \text{ for } 0 < x < \infty.
\end{aligned}
$$

Solution (5 of 6)

Taking the Fourier sine transform with respect to x results in a transformed general solution of the form,

$$\hat{u}_2^s(\omega, y) = A(\omega)e^{-\omega y} + B(\omega)e^{\omega y}.$$

If $\omega > 0$ and the solution is to remain bounded and asymptotically approach zero as $y \to \infty$, then $B(\omega) = 0$ for all ω. Taking the partial derivative of $\hat{u}_2^s(\omega, y)$ with respect to y, evaluating this partial derivative at $y = 0$, and setting it equal to the Fourier sine transform of the nonhomogeneous Neumann boundary condition results in

$$\begin{aligned} -\omega A(\omega) &= \mathcal{F}_s[e^{-x^2} \sin x](\omega) \\ A(\omega) &= \frac{-1}{\omega\sqrt{2}} e^{-\frac{\omega^2+1}{4}} \sinh \frac{\omega}{2} \end{aligned}$$

The solution $u_2(x, y)$ is the inverse Fourier sine transform of the product of this expression and $e^{-\omega y}$.

$$u_2(x, y) = \mathcal{F}_s^{-1}[A(\omega)e^{-\omega y}](x) = \frac{-1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\omega} e^{-\frac{\omega^2+4y\omega+1}{4}} \sinh \frac{\omega}{2} \sin(\omega x)\, d\omega$$

Solution (6 of 6)

The solution to the original boundary value problem is assembled as

$$
\begin{aligned}
u(x, y) &= u_1(x, y) + u_2(x, y) \\
&= \frac{1}{\pi} \int_0^\infty \frac{4 - 2\omega^2}{4 + \omega^4} e^{-\omega x} \cos(\omega y)\, d\omega \\
&\quad - \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\omega} e^{-\frac{\omega^2 + 4y\omega + 1}{4}} \sinh \frac{\omega}{2} \sin(\omega x)\, d\omega.
\end{aligned}
$$