Fourier Transform in Higher Dimensions

Partial Differential Equations

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Objectives

In this lesson we will:

- extend the definitions of the Fourier transform and its inverse to multidimensional settings,
- explore the properties of the extended Fourier transform, and
- use the Fourier transform in several examples.

Multidimensional Fourier Transform

Definition

If $f(x_1, x_2, \ldots, x_n)$ is absolutely integrable over $\mathbb{R}^n$ then the multidimensional Fourier transform of f is defined as

$$\begin{aligned}\hat{f}(\boldsymbol{\omega}) &= \mathcal{F}[f](\boldsymbol{\omega}) \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) e^{-i\omega_1 x_1 - \cdots - i\omega_n x_n}\, dx_1 \cdots dx_n.\end{aligned}$$

The inverse transform is

$$\begin{aligned}f(\boldsymbol{x}) &= \mathcal{F}^{-1}[\hat{f}](\boldsymbol{x}) \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\mathbb{R}^n} \hat{f}(\omega_1, \ldots, \omega_n) e^{i\omega_1 x_1 + \cdots + i\omega_n x_n}\, d\omega_1 \cdots d\omega_n.\end{aligned}$$

Vector Notation

Adopting the vector notation enables the transform and its inverse to be written concisely as follows

$$\hat{f}(\omega) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-i\omega\cdot\boldsymbol{x}}\, d\boldsymbol{x}$$

$$f(\boldsymbol{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\mathbb{R}^n} \hat{f}(\omega) e^{i\omega\cdot\boldsymbol{x}}\, d\omega.$$

The expression $\omega \cdot \boldsymbol{x}$ is the dot product or inner product of the vectors ω and $\boldsymbol{x}$.

Example

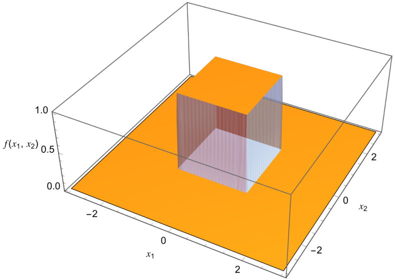
Find the Fourier transform of the function

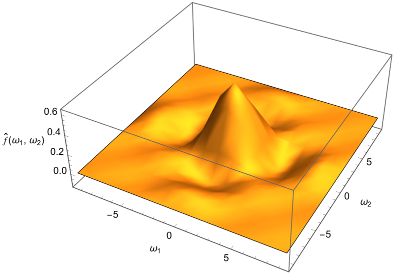
$$f(x_1, x_2) = \begin{cases} 1 & \text{for } (x_1, x_2) \in (-1, 1) \times (-1, 1) \\ 0 & \text{otherwise.} \end{cases}$$

Solution

$$\begin{aligned}\hat{f}(\omega_1,\omega_2) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2)e^{-i\omega_1 x_1 - i\omega_2 x_2}\,dx_1\,dx_2 \\ &= \frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1} e^{-i\omega_1 x_1 - i\omega_2 x_2}\,dx_1\,dx_2 \\ &= \frac{1}{2\pi}\int_{-1}^{1} e^{-i\omega_1 x_1}\,dx_1 \int_{-1}^{1} e^{-i\omega_2 x_2}\,dx_2 \\ &= \frac{2}{\pi}\frac{\sin\omega_1 \sin\omega_2}{\omega_1\omega_2}.\end{aligned}$$

Illustration





The surface on the left is the graph of the two-dimensional unitstep function and the surface on the right is the graph of its Fourier transform in the (ω_1, ω_2)-plane.

Shifting Property

Theorem

Suppose function $f(\boldsymbol{x})$ has Fourier transform $\hat{f}(\omega)$ and that $\boldsymbol{a}$ is a constant vector in $\mathbb{R}^n$, then

$$\mathcal{F}[f(\boldsymbol{x}-\boldsymbol{a})](\omega) = e^{-i\omega\cdot\boldsymbol{a}}\hat{f}(\omega)$$

$$\mathcal{F}[f(\boldsymbol{x})e^{-i\boldsymbol{a}\cdot\boldsymbol{x}}](\omega) = \hat{f}(\omega+\boldsymbol{a}).$$

Derivative Property

Theorem

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is absolutely integrable over $\mathbb{R}^n$, that all necessary partial derivatives of f are as well, and that $|f(\boldsymbol{x})|$ and its partial derivatives vanish as $\|\boldsymbol{x}\| \to \infty$.

$$\mathcal{F}\left[\frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial x_1^{k_1}\partial x_2^{k_2}\cdots\partial x_n^{k_n}}[f(\boldsymbol{x})]\right](\boldsymbol{\omega}) = i^{k_1+k_2+\cdots+k_n}\omega_1^{k_1}\omega_2^{k_2}\cdots\omega_n^{k_n}\hat{f}(\boldsymbol{\omega})$$

Fourier Convolution

This Fourier convolution is defined as

$$(f * g)(\boldsymbol{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\mathbb{R}^n} f(\boldsymbol{x} - \boldsymbol{z})g(\boldsymbol{z})\, d\boldsymbol{z}.$$

Theorem (Fourier Convolution Theorem)

Suppose functions f and g have Fourier transforms $\hat{f}$ and $\hat{g}$ respectively, then

$$\mathcal{F}[f * g](\omega) = \hat{f}(\omega)\hat{g}(\omega),$$

in other words the Fourier transform of the Fourier convolution of f and g is the product of the Fourier transforms of f and g.

Application: Newton's Theory of Gravitation

Let $\boldsymbol{x} \in \mathbb{R}^3$ and suppose $\phi(\boldsymbol{x})$ is the gravitational potential at position $\boldsymbol{x}$. The partial differential equation modeling this situation is Poisson's equation taking the form:

$$\Delta\phi(\boldsymbol{x}) = -4\pi G\rho(\boldsymbol{x}).$$

The notation $\Delta\phi(\boldsymbol{x})$ is a shorthand for $\phi_{xx} + \phi_{yy} + \phi_{zz}$. The constant G is the universal gravitational constant. The function $\rho(\boldsymbol{x})$ is the mass density at position $\boldsymbol{x}$.

Fourier Transform

$$
\begin{aligned}
\phi_{xx} + \phi_{yy} + \phi_{zz} &= -4\pi G \rho(\boldsymbol{x}) \\
\mathcal{F}[\phi_{xx} + \phi_{yy} + \phi_{zz}](\omega) &= -4\pi G \mathcal{F}[\rho(\boldsymbol{x})](\omega) \\
-(\omega_1^2 + \omega_2^2 + \omega_3^2)\hat{\phi}(\boldsymbol{\omega}) &= -4\pi G \hat{\rho}(\boldsymbol{\omega}) \\
\hat{\phi}(\boldsymbol{\omega}) &= \frac{4\pi G}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}} \hat{\rho}(\boldsymbol{\omega})
\end{aligned}
$$

Inverse Fourier Transform

$$\hat{\phi}(\boldsymbol{\omega}) = \frac{4\pi G}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}} \hat{\rho}(\boldsymbol{\omega})$$

$$\phi(\boldsymbol{x}) = \left(\frac{1}{\sqrt{2\pi}} \right)^3 \int_{\mathbb{R}^3} \frac{4\pi G}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}} \hat{\rho}(\boldsymbol{\omega}) e^{i\boldsymbol{\omega} \cdot \boldsymbol{x}} \, d\boldsymbol{\omega}$$

Example: Heat Equation

Consider the initial value problem for heat diffusion in three-dimensional space.

$$
\begin{aligned}
u_t &= \kappa \Delta u \text{ for } \boldsymbol{x} \in \mathbb{R}^3 \text{ and } t > 0 \\
u(\boldsymbol{x}, 0) &= \delta(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \mathbb{R}^3
\end{aligned}
$$

Solution (1 of 3)

The Fourier transform of the partial differential equation results in the ordinary differential equation,

$$\hat{u}_t(\boldsymbol{\omega}, t) = -\kappa(\omega_1^2 + \omega_2^2 + \omega_3^2)\hat{u}(\boldsymbol{\omega}, t).$$

This is a first-order, linear ordinary differential equation whose general solution is

$$\hat{u}(\boldsymbol{\omega}, t) = \hat{u}(\boldsymbol{\omega}, 0)e^{-\kappa(\boldsymbol{\omega}\cdot\boldsymbol{\omega})t}.$$

When $t = 0$ the left-hand side of this equation will equal the Fourier transform of the Dirac delta function.

$$\hat{u}(\boldsymbol{\omega}, 0) = \frac{1}{(2\pi)^{3/2}}$$

Solution (2 of 3)

$$\hat{u}(\omega, t) = \frac{1}{(2\pi)^{3/2}} e^{-\kappa(\boldsymbol{\omega}\cdot\boldsymbol{\omega})t}.$$

Inverse transform this expression.

$$\begin{aligned}
u(\boldsymbol{x}, t) &= \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int_{\mathbb{R}^3} \frac{1}{(2\pi)^{3/2}} e^{-\kappa(\boldsymbol{\omega}\cdot\boldsymbol{\omega})t} e^{i\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{\omega} \\
&= \left(\frac{1}{2\pi}\right)^3 e^{-\frac{\boldsymbol{x}\cdot\boldsymbol{x}}{4\kappa t}} \int_{\mathbb{R}^3} e^{-\kappa t\left(\omega_1 - \frac{ix_1}{2\kappa t}\right)^2} e^{-\kappa t\left(\omega_2 - \frac{ix_2}{2\kappa t}\right)^2} e^{-\kappa t\left(\omega_3 - \frac{ix_3}{2\kappa t}\right)^2}\, d\boldsymbol{\omega}
\end{aligned}$$

Solution (3 of 3)

$$u(\boldsymbol{x}, t) = \left(\frac{1}{2\pi}\right)^3 e^{-\frac{\boldsymbol{x}\cdot\boldsymbol{x}}{4\kappa t}} \int_{\mathbb{R}^3} e^{-\kappa t\left(\omega_1 - \frac{ix_1}{2\kappa t}\right)^2} e^{-\kappa t\left(\omega_2 - \frac{ix_2}{2\kappa t}\right)^2} e^{-\kappa t\left(\omega_3 - \frac{ix_3}{2\kappa t}\right)^2} d\boldsymbol{\omega}$$

$$U(\boldsymbol{x}, t) = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-\frac{\boldsymbol{x}\cdot\boldsymbol{x}}{4\kappa t}}$$

Remark: this is the fundamental solution to the heat equation in three dimensions.