Fourier Transform and its Properties

Partial Differential Equations

J Robert Buchanan

Department of Mathematics

Fall 2025

Objectives

In this lesson we will:

- define the Fourier transform,
- define the inverse Fourier transform,
- calculate the Fourier transform and inverse Fourier transform of some common functions,
- state some properties of the Fourier transform, and
- define the Fourier convolution.

Definition of Fourier Transform

Definition

Let $f(x)$ be defined for all real numbers. The Fourier transform of f is denoted by $\mathcal{F}[f](\omega)$ or often as $\hat{f}(\omega)$. The Fourier transform of f is

$$\mathcal{F}[f](\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx,$$

where $i = \sqrt{-1}$. The Fourier transform of f is meaningful if and only if the improper integral converges.

Remarks:

- The Fourier transform maps the original function f of variable x to a new function $\hat{f}$ of variable ω which is often thought of as a frequency.
- There are other definitions of the Fourier Transform that differ only in a multiplicative constant.

Existence of Fourier Transform

Sufficient conditions for the existence of the Fourier transform are the following,

- $\displaystyle\int_{-\infty}^{\infty} |f(x)|\, dx$ converges,
- f and f' are piecewise continuous on every interval of the form $[-M, M]$ for arbitrary $M > 0$.

The first condition is known as **absolute integrability** on the real number line.

Example

Find the Fourier transform of the piecewise-defined function

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Solution

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega x}\, dx \\
&= -\frac{1}{i\omega\sqrt{2\pi}} \left(e^{-i\omega} - e^{i\omega} \right) \\
&= \frac{\sqrt{2}}{\omega\sqrt{\pi}} \sin\omega
\end{aligned}$$

Definition of Inverse Fourier Transform

Definition

The inverse Fourier transform of $\hat{f}(\omega)$ is given by

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega,$$

provided the improper integral converges.

Remarks:

- The original function can be recovered from its Fourier transform.
- If $f(x)$ is absolutely integrable and piecewise smooth, then

$$\frac{1}{2}\left(f(x-) + f(x+)\right) = \mathcal{F}^{-1}[\hat{f}](x).$$

Therefore, at all points x where f is continuous, $f(x) = \mathcal{F}^{-1}[\hat{f}](x)$.

Example

Suppose a is a positive constant, and find the inverse Fourier transform of $e^{-a|w|}$.

Solution

$$
\begin{aligned}
\mathcal{F}^{-1}[e^{-a|w|}](x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|w|} e^{i\omega x}\, d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a+ix)\omega}\, d\omega + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{(-a+ix)\omega}\, d\omega \\
&= \frac{1}{(a+ix)\sqrt{2\pi}} + \frac{1}{(a-ix)\sqrt{2\pi}} \\
&= \frac{a\sqrt{2}}{(a^2+x^2)\sqrt{\pi}}.
\end{aligned}
$$

Linear Transformation

Theorem

Let $f(x)$ and $g(x)$ be functions having Fourier transforms $\hat{f}(\omega)$ and $\hat{g}(\omega)$ and let a and b be any complex numbers, then $af(x) + bg(x)$ has a Fourier transform and

$$\mathcal{F}[af + bg](\omega) = a\hat{f}(\omega) + b\hat{g}(\omega).$$

Fourier Transform of a Derivative

Theorem

If $f(x)$, $f'(x)$, ..., $f^{(n-1)}(x)$ are all Fourier transformable, $\lim_{x \to \pm\infty} f^{(k)}(x) = 0$ *for* $k = 0, 1, \ldots, n-1$, *and if* $f^{(n)}(x)$ *exists then* $\mathcal{F}[f^{(n)}](\omega) = (i\omega)^n \hat{f}(\omega)$.

Derivative of a Fourier Transform

Theorem

If function $f(x)$ is Fourier transformable and such that the operations of integration and differentiation can be interchanged, then

$$\mathcal{F}[x^n f(x)](\omega) = i^n \frac{d^n}{d\omega^n}\left[\hat{f}(\omega)\right].$$

Example

Find the Fourier transform of the “sign” function,

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Solution

The sign function is not absolutely integrable, but the Dirac delta function and the previous theorem can overcome this drawback. Making use of the integral,

$$-1 + 2\int_{-\infty}^{x} \delta(\tau)\, d\tau = \operatorname{sgn}(x).$$

Thus $\operatorname{sgn}'(x) = 2\delta(x)$. Therefore,

$$i\omega\mathcal{F}[\operatorname{sgn}](\omega) = \mathcal{F}[\operatorname{sgn}'](\omega) = \mathcal{F}[2\delta](\omega) = \frac{2}{\sqrt{2\pi}} \implies \mathcal{F}[\operatorname{sgn}](\omega) = \frac{2}{i\omega\sqrt{2\pi}}.$$

Example

Find the Fourier transform of the Heaviside function,

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Solution

The Heaviside function can be built from the sign function of the previous example.

$$H(x) = \frac{1}{2}(1 + \operatorname{sgn}(x))$$

This implies,

$$\mathcal{F}[H](\omega) = \frac{1}{2}\mathcal{F}[1](\omega) + \frac{1}{2}\mathcal{F}[\operatorname{sgn}](\omega) = \frac{1}{2}\mathcal{F}[1](\omega) + \frac{1}{i\omega\sqrt{2\pi}}.$$

The constant function 1 is not absolutely integrable, so instead find a Fourier transform $\hat{f}(\omega)$ whose inverse Fourier transform is 1.

$$1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega \implies \hat{f}(\omega) = \sqrt{2\pi}\delta(\omega)$$

Together these yield,

$$\mathcal{F}[H](\omega) = \sqrt{\frac{\pi}{2}}\delta(\omega) - \frac{i}{\omega\sqrt{2\pi}}.$$

Shifting Property

Theorem

Suppose function f has Fourier transform $\hat{f}$ and that a is a constant, then $f(x-a)$ and $f(x)e^{-iax}$ have Fourier transforms and

$$\mathcal{F}[f(x-a)](\omega) = e^{-i\omega a}\hat{f}(\omega)$$
$$\mathcal{F}[f(x)e^{-iax}](\omega) = \hat{f}(\omega+a).$$

Fourier Convolution

Definition

The **Fourier convolution** of two functions f and g is

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-z)g(z)\,dz,$$

provided the improper integral converges.

Theorem (Fourier Convolution Theorem)

Suppose functions f and g have Fourier transforms $\hat{f}$ and $\hat{g}$ respectively, then $(f * g)(x)$ has a Fourier transform and

$$\mathcal{F}[f * g](\omega) = \hat{f}(\omega)\hat{g}(\omega).$$

Remark: the Fourier transform of the Fourier convolution of f and g is the product of the Fourier transforms of f and g.