Solving PDEs Using the Fourier Transform

Partial Differential Equations

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Objectives

In this lesson we will:

- use the Fourier transform and inverse Fourier transform to solve various partial differential equations.

Solving the Unbounded Heat Equation

Consider the heat equation with its spatial domain as the entire real number line.

$$
\begin{aligned}
u_t &= \kappa u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\
u(x,0) &= e^{-x^2} \sin x \text{ for } -\infty < x < \infty
\end{aligned}
$$

Apply Fourier Transform

$$\begin{aligned} u_t &= \kappa u_{xx} \\ \mathcal{F}[u_t(x,t)](\omega) &= \kappa \mathcal{F}[u_{xx}(x,t)](\omega) \\ \hat{u}_t(\omega,t) &= \kappa (i\omega)^2 \hat{u}(\omega,t) \\ \hat{u}_t(\omega,t) &= -\kappa\omega^2 \hat{u}(\omega,t) \end{aligned}$$

This is now a first-order, linear ordinary differential equation with unknown function $\hat{u}(\omega, t)$.

Solve the Ordinary Differential Equation

$$\hat{u}_t(\omega, t) = -\kappa\omega^2 \hat{u}(\omega, t)$$

$$\hat{u}(\omega, t) = \hat{u}(\omega, 0) e^{-\kappa\omega^2 t}$$

where $\hat{u}(\omega, 0)$ is the Fourier transform of the initial condition $u_0(x)$.

$$\hat{u}(\omega, 0) = \frac{i}{2\sqrt{2}} \left(e^{-(\omega+1)^2/4} - e^{-(\omega-1)^2/4} \right)$$

Apply Inverse Fourier Transform

$$\hat{u}(\omega, t) = \frac{i}{2\sqrt{2}} \left(e^{-(\omega+1)^2/4} - e^{-(\omega-1)^2/4} \right) e^{-\kappa\omega^2 t}$$

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega, t) e^{i\omega x}\, d\omega \\ &= \frac{1}{\sqrt{1 + 4\kappa t}} e^{-\frac{x^2 + \kappa t}{1 + 4\kappa t}} \sin\left(\frac{x}{1 + 4\kappa t} \right) \end{aligned}$$

Solving the Wave Equation

Consider the initial value problem for the unbounded string.

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\
u(x,0) &= f(x) \text{ for } -\infty < x < \infty \\
u_t(x,0) &= g(x) \text{ for } -\infty < x < \infty
\end{aligned}
$$

Solve this initial value problem using the Fourier transform.

Apply Fourier Transform

$$
\begin{aligned}
\mathcal{F}[u_{tt}](\omega) &= c^2 \mathcal{F}[u_{xx}](\omega) \\
\hat{u}_{tt}(\omega, t) &= -c^2 \omega^2 \hat{u}(\omega, t) \\
\hat{u}_{tt}(\omega, t) + c^2 \omega^2 \hat{u}(\omega, t) &= 0
\end{aligned}
$$

This ordinary differential equation models simple harmonic motion without damping.

Solve the Ordinary Differential Equation

$$\hat{u}_{tt}(\omega, t) + c^2\omega^2\hat{u}(\omega, t) = 0$$

$$\hat{u}(\omega, t) = \hat{A}e^{-ic\omega t} + \hat{B}e^{ic\omega t},$$

where $\hat{A}$ and $\hat{B}$ are functions of ω but constant with respect to variable t.

Apply Inverse Fourier Transform

$$\hat{u}(\omega, t) = \hat{A}e^{-ic\omega t} + \hat{B}e^{ic\omega t}$$

$$\mathcal{F}^{-1}[\hat{u}(\omega, t)](x) = \mathcal{F}^{-1}[\hat{A}(\omega)e^{-ic\omega t} + \hat{B}(\omega)e^{ic\omega t}](x)$$

$$u(x, t) = A(x - ct) + B(x + ct)$$

Note that the inverse Fourier transform made use of the shifting property of the Fourier transform

Apply the Initial Conditions

$$
\begin{aligned}
u(x,t) &= A(x-ct) + B(x+ct) \\
u(x,0) &= A(x) + B(x) = f(x) \\
u_t(x,0) &= -cA'(x) + cB'(x) = g(x).
\end{aligned}
$$

Integrating the second equation from 0 to x yields,

$$
-A(x) + B(x) = \frac{1}{c}\int_0^x g(s)\,ds.
$$

Solving for $A(x)$ and $B(x)$ yields

$$
\begin{aligned}
A(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds \\
B(x) &= \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds.
\end{aligned}
$$

Thus we find the d'Alembertian solution.

$$
u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds
$$

Solving Laplace's Equation

Consider the boundary value problem below.

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0 \text{ for } -\infty < x < \infty \text{ and } y > 0 \\
u(x,0) &= \frac{1}{1+x^2} \text{ for } -\infty < x < \infty \\
\lim_{x\to\infty} u(x,y) &= 0 \text{ for } y > 0 \\
\lim_{x\to-\infty} u(x,y) &= 0 \text{ for } y > 0 \\
\lim_{y\to\infty} u(x,y) &= 0 \text{ for } -\infty < x < \infty
\end{aligned}
$$

Note that the boundary conditions at infinity force the solution to be bounded.

Apply Fourier Transform

The Fourier transform can be applied to the partial differential equation with respect to x.

$$\mathcal{F}[u_{xx}(x,y) + u_{yy}(x,y)](\omega) = -\omega^2 \hat{u}(\omega, y) + \hat{u}_{yy}(\omega, y) = 0$$

Solve the Ordinary Differential Equation

$$\hat{u}_{yy}(\omega, y) - \omega^2 \hat{u}(\omega, y) = 0$$
$$\hat{u}(\omega, y) = \hat{A}(\omega) e^{\omega y} + \hat{B}(\omega) e^{-\omega y}$$

Functions $\hat{A}$ and $\hat{B}$ are constant with respect to y.

Apply Boundary Conditions

The boundary condition, $\lim_{y\to\infty} u(x,y) = 0$ implies $\lim_{y\to\infty} \hat{u}(\omega,y) = 0$. Thus when $\omega > 0$, $\hat{A}(\omega) = 0$ and when $\omega < 0$, $\hat{B}(\omega) = 0$. Hence the Fourier transform of the solution may be written as a piecewise-defined function,

$$\hat{u}(\omega,y) = \begin{cases} \hat{A}(\omega)e^{\omega y} & \text{if } \omega < 0 \\ \hat{B}(\omega)e^{-\omega y} & \text{if } \omega > 0. \end{cases}$$

The boundary condition along the x-axis implies

$$\hat{u}(\omega,0) = \mathcal{F}\left[\frac{1}{1+x^2}\right](\omega) = \frac{\sqrt{\pi}e^{-|\omega|}}{\sqrt{2}}.$$

Combining this condition with the piecewise definition of $\hat{u}(\omega,y)$ when $y = 0$ implies

$$\hat{A}(\omega) = \begin{cases} \frac{\sqrt{\pi}e^{\omega}}{\sqrt{2}} & \text{if } \omega < 0 \\ 0 & \text{if } \omega > 0 \end{cases}$$

$$\hat{B}(\omega) = \begin{cases} 0 & \text{if } \omega < 0 \\ \frac{\sqrt{\pi}e^{-\omega}}{\sqrt{2}} & \text{if } \omega > 0. \end{cases}$$

Apply Inverse Fourier Transform

$$\hat{u}(\omega, y) = \frac{\sqrt{\pi} e^{-(1+y)|\omega|}}{\sqrt{2}}$$

$$u(x, y) = \mathcal{F}^{-1}\left[\frac{\sqrt{\pi} e^{-(1+y)|\omega|}}{\sqrt{2}}\right](x) = \frac{1+y}{x^2 + (1+y)^2}$$