Green's Functions for General Boundary Value Problems

Partial Differential Equations

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Objectives

In this lesson we will:

- present the general nth-order linear differential operator,
- present the general nth-order linear boundary value problem,
- describe the properties of a Green's function in the setting of a general nth-order linear boundary value problem,
- use a Green's function to solve an nth-order linear boundary value problem,
- introduce functionals, distributions, and symbolic differentiation.

nth-Order Boundary Value Problem

$$L[\cdot] = a_n(x)\frac{d^n}{dx^n}[\cdot] + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}[\cdot] + \cdots + a_1(x)\frac{d}{dx}[\cdot] + a_0(x)[\cdot]$$

where the coefficient functions $a_i(x)$ for $i = 0, 1, \ldots, n$ are defined and continuous on the interval $[a, b]$ with $a_n(x) \neq 0$ for $a \leq x \leq b$. $\mathcal{D}(L)$ will be the functions $u(x)$ with piecewise continuous nth derivatives on $[a, b]$. An nth order linear boundary value problem is

$$\begin{aligned} L[u] &= -f \text{ for } a < x < b \\ B_1[u] &= \sum_{j=0}^{n-1}(a_{1,j}u^{(j)}(a) + b_{1,j}u^{(j)}(b)) = 0 \\ &\vdots \\ B_n[u] &= \sum_{j=0}^{n-1}(a_{n,j}u^{(j)}(a) + b_{n,j}u^{(j)}(b)) = 0 \end{aligned}$$

where $a_{i,j}$ and $b_{i,j}$ for $i = 1, \ldots, n$ and $j = 0, \ldots, n-1$ are constants and B_i denotes a linear combination of function $u(x)$ and its derivatives through order $n-1$. The boundary conditions are assumed to be linearly independent. In boundary condition $B_k[u]$ at least one of $a_{k,j}$ or $b_{k,j}$ is nonzero for all $k = 1, 2, \ldots, n$.

Properties of Green's Function

By definition the Green's function $G(x; y)$,

- satisfies the ordinary differential equation $L[G] = 0$ for $(x, y) \in (a, b) \times (a, b)$ when $x \neq y$, $G(x; y)$ and its derivatives with respect to x up to order n exist and are continuous for $a < x < y$ and $y < x < b$.
- $G(x; y)$ satisfies the boundary conditions of the BVP.
- For $k = 0, 1, \ldots, n-2$ the derivative $\frac{\partial^k}{\partial x^k}[G(x; y)]$ is continuous across the line where $x = y$, in other words

$$\lim_{x \to y^+} \frac{\partial^k}{\partial x^k}[G(x; y)] - \lim_{x \to y^-} \frac{\partial^k}{\partial x^k}[G(x; y)] = 0 \text{ for } k = 0, 1, \ldots, n-2.$$

- $\frac{\partial^{n-1}}{\partial x^{n-1}}[G(x; y)]$ has a jump discontinuity crossing the line $x = y$, specifically

$$\lim_{x \to y^+} \frac{\partial^{n-1}}{\partial x^{n-1}}[G(x; y)] - \lim_{x \to y^-} \frac{\partial^{n-1}}{\partial x^{n-1}}[G(x; y)] = \frac{-1}{a_n(y)}.$$

Existence and Uniqueness

Theorem

If the homogeneous boundary value problem of the nth-order boundary value problem (that is, the case where $f(x) = 0$ *on* (a, b)*) has only the zero solution, then there exists a unique Green's function* $G(x; y)$ *associated with the boundary value problem.*

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Theorem

If the homogeneous boundary value problem of (again, when $f(x) = 0$ *on* (a, b)*) has only the zero solution, then the solution to the nonhomogeneous form can be expressed as the integral*

$$u(x) = \int_a^b G(x; y) f(y)\, dy.$$

where $G(x; y)$ *is the Green's function of the associated homogeneous boundary value problem.*

Example

Use the Green's function approach to find the solution to the following boundary value problem.

$$u'''(x) = e^x \text{ for } 0 < x < 1$$
$$u(0) = u'(0) = u''(1) = 0$$

Solution (1 of 5)

Green's function must satisfy $G_{xxx}(x;y) = 0$ on $[0,1] \times [0,1]$ where $x \neq y$. The general form of the Green's function is

$$G(x;y) = \begin{cases} \alpha_1(y) + \alpha_2(y)x + \alpha_3(y)x^2 & \text{if } 0 \leq x < y \leq 1 \\ \beta_1(y) + \beta_2(y)x + \beta_3(y)x^2 & \text{if } 0 \leq y < x \leq 1. \end{cases}$$

The coefficients must be chosen to satisfy the boundary conditions, the continuity of $G(x;y)$ and $G_x(x;y)$ on $[0,1] \times [0,1]$, and the jump discontinuity of $G_{xx}(x;y)$ at $x = y$.

Solution (2 of 5)

The first partial derivative of $G(x; y)$ with respect to x can be expressed as

$$G_x(x; y) = \begin{cases} \alpha_2(y) + 2\alpha_3(y)x & \text{if } 0 \leq x < y \leq 1 \\ \beta_2(y) + 2\beta_3(y)x & \text{if } 0 \leq y < x \leq 1. \end{cases}$$

The boundary condition $G_x(0; y) = 0$ implies $\alpha_2(y) = 0$. The second partial derivative of $G(x; y)$ with respect to x can be expressed as

$$G_{xx}(x; y) = \begin{cases} 2\alpha_3(y) & \text{if } 0 \leq x < y \leq 1 \\ 2\beta_3(y) & \text{if } 0 \leq y < x \leq 1. \end{cases}$$

The boundary condition $G_{xx}(1; y) = 0$ implies $\beta_3(y) = 0$.

Solution (3 of 5)

$G(x; y)$ now has the simplified form of

$$G(x;y) = \begin{cases} \alpha_3(y)x^2 & \text{if } 0 \leq x < y \leq 1 \\ \beta_1(y) + \beta_2(y)x & \text{if } 0 \leq y < x \leq 1. \end{cases}$$

Continuity of $G(x; y)$ on $(0, 1) \times (0, 1)$ implies

$$\lim_{x \to y^-} G(x;y) = \lim_{x \to y^+} G(x;y) \iff \alpha_3(y)y^2 = \beta_1(y) + \beta_2(y)y$$

and similarly continuity of $G_x(x; y)$ on $(0, 1) \times (0, 1)$ implies

$$\lim_{x \to y^-} G_x(x;y) = \lim_{x \to y^+} G_x(x;y) \iff 2\alpha_3(y)y = \beta_2(y).$$

Solution (4 of 5)

The jump discontinuity in $G_{xx}(x; y)$ across the line $x = y$ requires

$$G_{xx}(y+; y) - G_{xx}(y-; y) = \frac{-1}{1} = -1 \iff 0 - 2\alpha_3(y) = -1$$
$$\iff \alpha_3(y) = \frac{1}{2}.$$

This results yields $\beta_2(y) = y$ and $\beta_1(y) = -\frac{1}{2}y^2$. Finally the Green's function is

$$G(x; y) = \begin{cases} \frac{1}{2}x^2 & \text{if } 0 \le x < y \le 1 \\ xy - \frac{1}{2}y^2 & \text{if } 0 \le y < x \le 1 \end{cases}$$

Solution (5 of 5)

$$
\begin{aligned}
u(x) &= \int_0^1 G(x;y)(-e^y)\,dy \\
&= \int_0^x \left(\frac{y^2}{2} - x\,y\right)(-e^y)\,dy + \int_x^1 \frac{-x^2}{2}(-e^y)\,dy \\
&= -1 - x - \frac{x^2}{2}e^x + e^x + \frac{x^2}{2}e^x - \frac{ex^2}{2} \\
&= -1 - x - \frac{e}{2}x^2 + e^x
\end{aligned}
$$

Symbolic Functions, Test Functions, and Functionals

- Symbolic functions are not required to have numerical values for all real number inputs.

Example

$\delta(x) = 0$ for $x \neq 0$, but $\delta(0)$ has no defined value.

- Test functions consist of ordinary functions which have continuous derivatives of all orders and which vanish outside of some closed and bounded interval. These functions are denoted C_0^∞ and are called functions with compact support.

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- Test functions consist of ordinary functions which have continuous derivatives of all orders and which vanish outside of some closed and bounded interval. These functions are denoted C_0^∞ and are called functions with compact support.
- A **functional** F is a mapping from the vector space of test functions to the real (or complex) numbers.
- A functional F defined on a vector space of test functions is called a **linear functional** if whenever ϕ_1 and ϕ_2 are test functions and k is a real or complex scalar, then $F(\phi_i)$ is a real or complex number and the following equation is satisfied.

$$F(\phi_1 + k\,\phi_2) = F(\phi_1) + k\,F(\phi_2)$$

Distributions

If $F(\phi) = \langle f, \phi \rangle$ is continuous, then it is referred to as a **distribution.** Every distribution has a derivative which is itself another distribution. For example, suppose f is a fixed, differentiable function from an inner product space, ϕ is a test function and

$$F(\phi) = \langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\, dx.$$

Using integration by parts,

$$\begin{aligned}\int_{-\infty}^{\infty} f'(x)\phi(x)\, dx &= [f(x)\phi(x)]_{x\to-\infty}^{x\to\infty} - \int_{-\infty}^{\infty} f(x)\phi'(x)\, dx \\ &= -\int_{-\infty}^{\infty} f(x)\phi'(x)\, dx.\end{aligned}$$

Thus the derivative of F (which will be denoted F') can be expressed as

$$F'(\phi) = -\langle f, \phi' \rangle = -\int_{-\infty}^{\infty} f(x)\phi'(x)\, dx.$$

Symbolic Differentiation

The Dirac delta function can be defined as the distribution with the property that $\langle \delta, \phi \rangle = \phi(0)$ for all test functions ϕ. The derivative of $\delta(x)$ is then the distribution for which

$$\int_{-\infty}^{\infty} \delta'(x)\phi(x)\,dx = -\phi'(0),$$

for all test functions. This process of determining the derivative of a distribution is called **symbolic differentiation**.

Inverse of a Linear Transformation

The inverse of a linear differentiation transformation L is an integral transformation with a kernel $G(x;y)$ of the form

$$L^{-1}[f] = \int_a^b G(x;y)f(y)\,dy.$$

If the kernel is a distribution and u and v are test functions satisfying the equation $L[u] = v$ then

$$u = L^{-1}L[u] = L^{-1}[v] = \int_a^b G(x;y)v(y)\,dy$$

$$L\,L^{-1}[v] = L\left[\int_a^b G(x;y)v(y)\,dy\right].$$

Since the linear transformation L performs differentiation with respect to variable x, the operator may be pulled inside the integral formally. This implies

$$v(x) = \int_a^b L[G(x;y)]v(y)\,dy.$$

If this equation holds for all test functions v then $L[G(x;y)] = \delta(y-x)$.

Example

Find the inverse of the second-order differentiation transformation $L[\cdot] \equiv \frac{d^2}{dx^2}[\cdot]$.

Solution (1 of 2)

The inverse of L is an integral operator with a kernel $G(x; y)$ satisfying

$$L[G(x;y)] = \frac{d^2}{dx^2}[G(x;y)] = \delta(y-x).$$

We can show

$$\frac{d}{dx}[G(x;y)] = \int \delta(y-x)\,dx = -H(y-x) + A(y)$$

where H is the Heaviside function and $A(y)$ is an arbitrary function of y.

$$G(x;y) = \int (-H(y-x) + A(y))\,dx = (y-x)H(y-x) + xA(y) + B(y)$$

where $B(y)$ is another arbitrary function of y. The arbitrary functions A and B can be determined from boundary conditions.

Solution (2 of 2)

Suppose we have boundary conditions $u(a) = u(b) = 0$, then the adjoint will be the operator L^* satisfying the equation

$$\int_a^b L[u](x)v(x)\,dx = \int_a^b u(x)L^*[v](x)\,dx.$$

Applying the operator L and integrating by parts yield

$$\begin{aligned}
\int_a^b L[u](x)v(x)\,dx &= \int_a^b u''(x)v(x)\,dx \\
&= [u'(x)v(x)]_{x=a}^{x=b} - \int_a^b u'(x)v'(x)\,dx \\
&= -[u(x)v'(x)]_{x=a}^{x=b} + \int_a^b u(x)v''(x)\,dx \\
&= \int_a^b u(x)L[v](x)\,dx.
\end{aligned}$$

Hence L is self-adjoint.