Green's Functions for Initial Value Problems

Partial Differential Equations

J Robert Buchanan

Department of Mathematics

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Objectives

In this lesson we will:

- develop the Green's function for the nth-order initial value problem,
- use Green's functions to solve nth-order initial value problems.

nth-Order Initial Value Problem

For $n \geq 2$ the general nth order initial value problem is assumed here to have the form,

$$u^{(n)}(t) + a_{n-1}(t)u^{(n-1)}(t) + \cdots + a_1(t)u'(t) + a_0(t)u(t) = f(t)$$
$$u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0.$$

The left-hand side of the ordinary differential equation will be denoted $L[u](t)$, another linear differential operator. If the coefficient functions $a_0(t)$, $a_1(t)$, $\ldots$, $a_{n-1}(t)$, and $f(t)$ are continuous on an interval, say $[0, T]$, then a unique solution to the IVP exists on that interval.

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Remarks:

- This is not the most general nth order IVP, but more general forms can usually be re-written in this form.
- The initial conditions are chosen to be all zero so that the solution to this IVP will be what is often called a “particular” solution.

Green's Function

A Green's function for the initial value problem will be

$$G(t;\tau) = \begin{cases} G_1(t;\tau) & \text{if } 0 \le t < \tau < \infty, \\ G_2(t;\tau) & \text{if } 0 \le \tau < t < \infty. \end{cases}$$

- $G(t;\tau)$ and its first $n-2$ partial derivatives with respect to t must be continuous on $(0,\infty) \times (0,\infty)$.

$$G_2(\tau;\tau) = \partial_t G_2(\tau;\tau) = \cdots = \partial_t^{n-2} G_2(\tau;\tau) = 0.$$

The $(n-1)$st partial derivative with respect to t is continuous on the region where $0 \le t < \tau < \infty$, is continuous on the region where $0 \le \tau < t < \infty$, and has a jump discontinuity of magnitude 1 across the line $t = \tau$, since

$$1 = \lim_{t \to \tau^+} \partial_t^{n-1} G(t;\tau) - \lim_{t \to \tau^-} \partial_t^{n-1} G(t;\tau) = \partial_t^{n-1} G(\tau;\tau) - 0 = \partial_t^{n-1} G(\tau;\tau).$$

Thus $L^*[G] = \delta(\tau - t)$.

Solution Using a Green's Function

The solution to the initial value problem is determined by an integral.

$$
\begin{aligned}
u(t) &= \int_0^\infty \delta(\tau - t) u(\tau)\, d\tau \\
&= \int_0^\infty L^*[G] u(\tau)\, d\tau \\
&= \int_0^\infty G(t; \tau) L[u](\tau)\, d\tau \\
&= \int_0^\infty G(t; \tau) f(\tau)\, d\tau
\end{aligned}
$$

Finding a Green's Function

There exists a fundamental set of solutions $\{u_1, u_2, \ldots, u_n\}$ such that

$$G(t;\tau) = C_1 u_1(t) + C_2 u_2(t) + \cdots + C_n u_n(t)$$

where the coefficients C_1, C_2, ..., C_n are functions of τ chosen so that $G(t;\tau)$ satisfies the continuity and jump conditions.

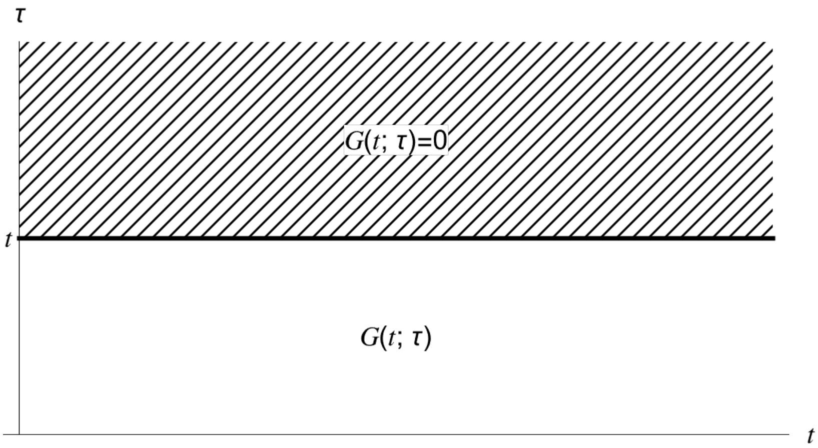
Once the Green's function is determined, the solution to the original initial value problem is found via an integral.

$$u(t) = \int_0^\infty G(t;\tau)f(\tau)\,d\tau = \int_0^t G(t;\tau)f(\tau)\,d\tau$$

The second equality holds since $G(t;\tau) = 0$ for $t < \tau$ by construction.

Causality Condition

The solution $u(t)$ depends only on the values of the nonhomogeneous forcing function f at times prior to t. The solution is said to obey a **causality condition** which agrees with physical intuition.



Example

Use the Green's function approach to solve the following initial value problem.

$$u'' - 4u = \cos t \text{ for } t > 0$$
$$u(0) = u'(0) = 0$$

Solution (1 of 2)

- On the interval $0 \leq \tau < t$ the Green's function is $G(t;\tau) = C_1 e^{2t} + C_2 e^{-2t}$ where the coefficients C_1 and C_2 are functions of τ.
- If $G(\tau;\tau) = 0$ then $C_1 e^{2\tau} = -C_2 e^{-2\tau}$.
- To satisfy the jump discontinuity requirement,

$$\begin{aligned} 2C_1 e^{2\tau} - 2C_2 e^{-2\tau} &= 1 \\ 2C_1 e^{2\tau} + 2C_1 e^{2\tau} &= 1 \\ C_1 &= \frac{1}{4} e^{-2\tau} \end{aligned}$$

  which implies $C_2 = \frac{-1}{4} e^{2\tau}$.
- The Green's function can be expressed as

$$G(t;\tau) = \begin{cases} 0 & \text{if } 0 \leq t < \tau < \infty, \\ -\frac{1}{2}\sinh(2(\tau - t)) & \text{if } 0 \leq \tau < t < \infty. \end{cases}$$

Solution (2 of 2)

The solution to the original initial value problem can be found as

$$\begin{aligned}
u(t) &= \int_0^\infty G(t;\tau)\cos\tau\, d\tau \\
&= -\frac{1}{2}\int_0^t \sinh(2(\tau - t))\cos\tau\, d\tau \\
&= \left[\frac{-1}{5}\cosh(2(\tau - t))\cos\tau - \frac{1}{10}\sinh(2(\tau - t))\sin\tau\right]_{\tau=0}^{\tau=t} \\
&= \frac{1}{5}\cosh(2t) - \frac{1}{5}\cos t.
\end{aligned}$$