# Green's Functions in Higher Dimensions

*Partial Differential Equations*

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# Objectives

In this lesson we will:

- introduce the second-order linear differential operator in two spatial dimensions,
- introduce a second-order linear differential operator in three spatial dimensions,
- present Green's identities,
- introduce the notion of the principal solution.

# Second-Order Linear BVP

$$
\begin{aligned}
L[u] &= Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = \Gamma \\
B[u] &= a\,u + b(\nabla u \cdot \mathbf{N}) = c
\end{aligned}
$$

- $A$, $B$, $C$, $\ldots$, $F$, and $\Gamma$ are sufficiently smooth functions of $x$ and $y$.
- The PDE is assumed to hold over some region $R$ in the $xy$-plane.
- The boundary conditions are defined on $\partial R$ where $\mathbf{N}$ is the unit outward normal vector to $\partial R$ at coordinates $(x, y)$.
- Coefficients $a$, $b$, and $c$ are permitted to be functions of $(x, y)$ along $\partial R$. If $c = 0$ the boundary conditions are homogeneous.

# Flux Form of Green's Theorem

## Theorem (Flux Form)

*Let $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ be a vector field where $M$ and $N$ are continuous and have continuous first partial derivatives on some open region $D \subset \mathbb{R}^2$. Let $R \subset D$ with a piecewise-smooth, simple closed boundary curve $\partial R$, then*

$$\oint_{\partial R} \mathbf{F} \cdot \mathbf{N} \, ds = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dA.$$

The expression $ds$ is the differential arc length along the boundary curve of region $R$ and $dA$ is the differential area element of region $R$.

# Adjoint Operator

Define the quantities,

$$M = vAu_x - (vA)_x u + vBu_y + vDu$$
$$N = -(vB)_x u + vCu_y - (vC)_y u + vEu,$$

then

$$\iint_R vL[u]\,dA = \oint_{\partial R} [M\mathbf{i} + N\mathbf{j}] \cdot \mathbf{N}\,ds + \iint_R L^*[v]u\,dA$$

endequation where

$$L^*[v] = (Av)_{xx} + (Bv)_{xy} + (Cv)_{yy} - (Dv)_x - (Ev)_y + Fv.$$

The Green's function approach will be to solve the partial differential equation

$$L^*[G](x;y) = \delta(x-\xi, y-\eta)$$

where $\delta$ is now the multivariable version of the Dirac delta function and to force the Green's function $G$ to satisfy boundary conditions such that the surface terms vanish.

# Laplacian Operator

$$L[u] = u_{xx} + u_{yy}$$

In this setting $A = C = 1$ and $B = D = E = F = 0$. Thus $L^*[v] = L[v]$.

The adjoint equation simplifies to the following.

$$\iint_R vL[u]\, dA = \oint_{\partial R} (u_{\mathbf{N}} v - u v_{\mathbf{N}})\, ds + \iint_R L^*[v] u\, dA$$

The notations $u_{\mathbf{N}}$ and $v_{\mathbf{N}}$ denote the directional derivative of $u$ and $v$ in the direction of the outward normal vector to the boundary of region $R$.

# Helmholtz Operator

$$L[u] = u_{xx} + u_{yy} + k^2 u$$

In this setting $A = C = 1$ and $B = D = E = 0$ and $F = k^2$. Thus $L^*[v] = L[v]$.

The adjoint equation simplifies to the following.

$$\iint_R vL[u]\,dA = \oint_{\partial R} (u_{\mathbf{N}} v - u v_{\mathbf{N}})\,ds + \iint_R L^*[v] u\,dA$$

The notations $u_{\mathbf{N}}$ and $v_{\mathbf{N}}$ denote the directional derivative of $u$ and $v$ in the direction of the outward normal vector to the boundary of region $R$.

# Second-Order Linear BVP

$$
\begin{aligned}
L[u] &= Au_{xx} + Bu_{yy} + Cu_{zz} + Du_x + Eu_y + Fu_z + Gu = H \\
B[u] &= au + b(\nabla u \cdot \mathbf{N}) = c
\end{aligned}
$$

is defined on a region $Q \subset \mathbb{R}^3$.

- The functions $A$ through $H$ depending on $(x, y, z)$ are sufficiently smooth.
- Functions $a$, $b$, and $c$ present in the boundary conditions also depend on $(x, y, z)$.
- Vector $\mathbf{N}$ is the unit outward normal vector to the boundary surface of region $Q$.

# Divergence Theorem

## Theorem (Divergence)

*Let $Q$ be a closed, bounded subset of $\mathbb{R}^3$ with a piecewise smooth boundary $\partial Q$ and let $\mathbf{F}(x, y, z)$ be a vector field with continuous first partial derivatives. Then,*

$$\iint_{\partial Q} \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_{Q} (\nabla \cdot \mathbf{F})\, dV.$$

- The Divergence Theorem is also known as Gauss's Theorem.
- The symbol $dV$ is the differential volume element.
- The symbol $\mathbf{N}$ is the unit outward normal vector to the surface of $Q$.
- The symbol $dS$ is the differential surface area element.

# Adjoint Operator

Define

$$M = vAu_x - (vA)_x u + vDu$$
$$N = vBu_y - (vB)_y u + vEu$$
$$P = vCu_z - (vC)_z u + vFu.$$

The adjoint equation is then

$$\iiint_Q vL[u]\,dV = \iint_{\partial Q} [M\mathbf{i} + N\mathbf{j} + P\mathbf{k}] \cdot \mathbf{N}\,dS + \iiint_Q L^*[v]u\,dV$$

where

$$L^*[v] = (Av)_{xx} + (Bv)_{yy} + (Cv)_{zz} - (Dv)_x - (Ev)_y - (Fv)_z + Gv.$$

# Green's Identities

Green's First Identity:

$$\iint_{\partial Q} (g\nabla f) \cdot \mathbf{N}\, dS = \iiint_Q (\nabla f) \cdot (\nabla g)\, dV + \iiint_Q g(\Delta f)\, dV.$$

Green's Second Identity:

$$\iint_{\partial Q} [(g\nabla f) - (f\nabla g)] \cdot \mathbf{N}\, dS = \iiint_Q [g(\Delta f) - f(\Delta g)]\, dV.$$

# Inner Product

Define the inner product for functions $f(x, y)$ and $g(x, y)$ defined on region $R \subset \mathbb{R}^2$ as

$$\langle f, g \rangle = \iint_R f(x, y) g(x, y) \, dA.$$

# Strategy

Denote the Green's function as $G(x, y; \xi, \eta)$ where $x$ and $y$ will be thought of as the coordinates of a particular point $(x, y) \in R$ and $\xi$ and $\eta$ will be variables of integration. Suppose a Green's function can be found so that

- the integrand of the surface terms vanishes, and
- $L^*[G] = \delta(\xi - x, \eta - y)$.

If function $v$ in the adjoint equation is replaced by the Green's function $G$, then

$$\iint_R L[u]G\, dA = \iint_R uL^*[G]\, dA$$

and thus

$$\iint_R G(x, y; \xi, \eta)\Gamma(\xi, \eta)\, dA = \iint_R u(\xi, \eta)\delta(\xi - x, \eta - y)\, dA = u(x, y).$$

# Principal Solution

Suppose the Green's function is the sum of two functions,

$$G(x, y; \xi, \eta) = U(x, y; \xi, \eta) + g(x, y; \xi, \eta).$$

- Function $U$, known as the **principal solution** satisfies the partial differential equation $L^*[U] = \delta(\xi - x, \eta - y)$ but need not satisfy the boundary conditions imposed on the Green's function.
- Function $g$ will satisfy the differential equation $L^*[g] = 0$ and the boundary conditions required of the Green's function.