Solving Elliptic PDEs Using Green's Functions

Partial Differential Equations

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Objectives

In this lesson we will:

- solve Poisson's equation in two dimensions,
- solve Poisson's equation in three dimensions,
- introduce the method of images, and
- solve Helmholtz's equation in two dimensions.

Poisson's Equation

Consider Poisson's equation on a domain $D \subset \mathbb{R}^2$ with boundary ∂D.

$$L[u] = \Delta u = \phi \text{ for } (x, y) \in D$$
$$u = f \text{ for } (x, y) \in \partial D$$

If G is the Green's function, then

$$\iint_D G\phi \, dA = \oint_{\partial D} (u_{\mathbf{N}} G - u G_{\mathbf{N}}) \, ds + \iint_D L^*[G] u \, dA.$$

Since $L^*[G] = \delta(\xi - x, \eta - y)$ then this equation simplifies further to

$$\iint_D G\phi \, dA = \oint_{\partial D} (u_{\mathbf{N}} G - u G_{\mathbf{N}}) \, ds + u(x, y).$$

The boundary conditions specified above are of the Dirichlet type so $u = f$ on ∂D. However, $u_{\mathbf{N}}$ is not specified on ∂D, so assume $G = 0$ on ∂D.

$$u(x, y) = \iint_D G\phi \, dA + \oint_{\partial D} f G_{\mathbf{N}} \, ds.$$

Principal Solution

$$G(x, y; \xi, \eta) = U(x, y; \xi, \eta) + g(x, y; \xi, \eta)$$

The principal solution must satisfy the partial differential equation,

$$\Delta U = U_{\xi\xi} + U_{\eta\eta} = \delta(\xi - x, \eta - y)$$

on D. This equation has a physical interpretation in the setting of the gravitational potential. Imagine a unit point mass is fixed at (x, y), then U is the gravitational potential at (ξ, η) induced by the point mass.

Polar Coordinates

Convert this partial differential equation to polar coordinates with the origin at (x, y).

$$\xi - x = r\cos\theta$$
$$\eta - y = r\sin\theta$$

Since the gravitational potential induced at (ξ, η) will depend only on the distance from (x, y) to (ξ, η) which is coordinate r, the Laplacian of U can be written as

$$U_{rr} + \frac{1}{r}U_r = \frac{1}{r}(rU_r)_r = \delta(r).$$

When $r > 0$ the right hand side is 0, so a general solution is of the form $U(r) = c_1 \ln r + c_2$.

Principal Solution at $r = 0$

Are there values of the constants c_1 and c_2 which permit the equation $\Delta U = \delta(r)$ to be satisfied?

Define $U_\epsilon(r) = c_1 \ln(r + \epsilon) + c_2$ for $\epsilon > 0$.

$$\Delta U_\epsilon = \frac{1}{r}\left[r(U_\epsilon)_r\right]_r = \frac{c_1 \epsilon}{r(r+\epsilon)^2}$$

Integrate this Laplacian over $\mathbb{R}^2$ and set the result equal to the integral of the Dirac delta function over $\mathbb{R}^2$.

$$\iint_{\mathbb{R}^2} \Delta U_\epsilon \, dA = \iint_{\mathbb{R}^2} \delta(\xi - x, \eta - y)\, dA$$

which implies $2\pi c_1 = 1$ or $c_1 = 1/(2\pi)$. Constant c_2 is arbitrary and for convenience is chosen to be zero.

Limit of the Principal Solution

If $r > 0$,

$$\Delta U(r) = \lim_{\epsilon \to 0^+} \Delta U_\epsilon(r) = \lim_{\epsilon \to 0^+} \frac{\epsilon}{2\pi r(r+\epsilon)^2} = 0.$$

When $r = 0$ the expression for ΔU_ϵ is indeterminate of the form $0/0$. Applying l'Hôpital's rule produces

$$\lim_{\epsilon \to 0^+} \frac{\epsilon}{2\pi r(r+\epsilon)^2} = \lim_{\epsilon \to 0^+} \frac{1}{4\pi r(r+\epsilon)} = \frac{1}{4\pi r^2}$$

which "blows up" like the Dirac delta function at $r = 0$. Hence the principal solution $U(r) = \frac{1}{2\pi} \ln r$.

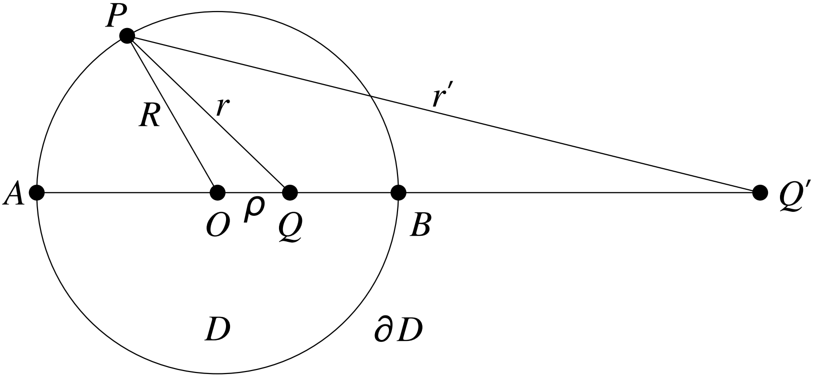
Hence we have,

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln r + g(x, y; \xi, \eta).$$

Completing the Green's Function

The function g must be chosen to satisfy the partial differential equation $L^*[g] = \Delta g = 0$ and $G = 0$ on the boundary of domain D. The solution is tied to the geometry of region D.

We will explore using the **method of images** to find the Green's function G on a disk of radius R.



Method of Images

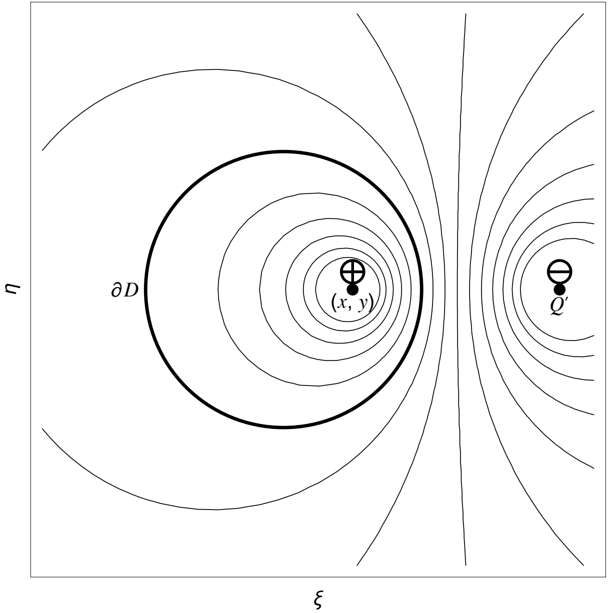
Suppose a unit point mass is placed at location Q (called the field point) and another "negative" unit mass is placed at Q' so that along the boundary of disk D the gravitational potential induced by the two masses vanishes. Such a potential will satisfy the requirements of a Green's function on the disk.

$$G = \frac{1}{4\pi} \ln \left(\frac{R^2}{\rho^2} \frac{r^2}{(r')^2} \right)$$

Using the change of variables $\xi = \hat{\rho} \cos \hat{\theta}$ and $\eta = \hat{\rho} \sin \hat{\theta}$ then

$$G = \frac{1}{4\pi} \ln \frac{R^2(\hat{\rho}^2 + \rho^2 - 2\hat{\rho}\rho \cos(\hat{\theta} - \theta))}{R^4 + \hat{\rho}^2\rho^2 - 2R^2\hat{\rho}\rho \cos(\hat{\theta} - \theta)}.$$

Illustration



Solution to Poisson's Equation on the Disk

Note that $\phi \equiv \phi(\hat{\rho}, \hat{\theta})$, $f \equiv f(\hat{\theta})$, $dA = \hat{\rho}\, d\hat{\rho}\, d\hat{\theta}$, and that $ds = R\, d\hat{\theta}$.

The directional derivative $G_{\mathbf{N}}$ where $\mathbf{N}$ is the unit outward normal vector to ∂D is the partial derivative of G with respect to $\hat{\rho}$ evaluated at $\hat{\rho} = R$.

$$u(\rho, \theta) = \int_0^{2\pi} \int_0^R G(\hat{\rho}, \hat{\theta}) \phi(\hat{\rho}, \hat{\theta}) \hat{\rho}\, d\hat{\rho}\, d\hat{\theta} + \int_0^{2\pi} f(\hat{\theta}) \left[\frac{\partial G}{\partial \hat{\rho}} \right]_{\hat{\rho}=R} R\, d\hat{\theta}.$$

The integral is often difficult to evaluate exactly, but can often be approximated numerically to any desired precision.

Example

Consider the Poisson boundary value problem,

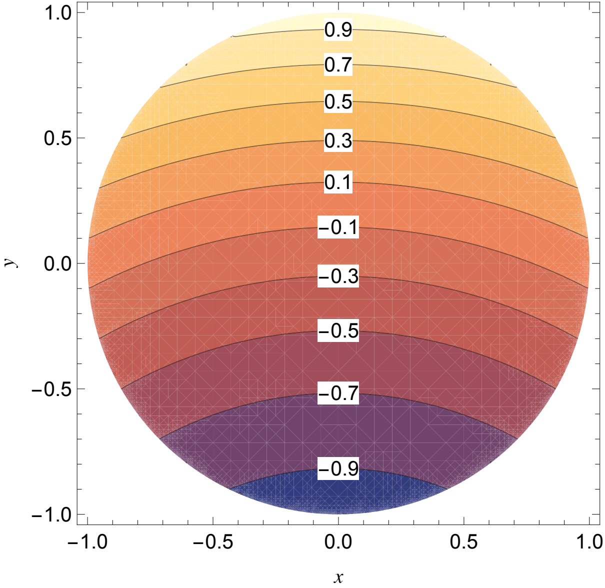
$$\begin{aligned} \Delta u &= 1 \text{ for } 0 \leq r < 1 \text{ and } 0 \leq \theta < 2\pi \\ u(1,\theta) &= \sin\theta \text{ for } 0 \leq \theta < 2\pi. \end{aligned}$$

Solution (1 of 2)

$$u(\rho,\theta) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 \hat{\rho}\ln\left(\frac{\hat{\rho}^2+\rho^2-2\hat{\rho}\rho\cos(\hat{\theta}-\theta)}{1+\hat{\rho}^2\rho^2-2\hat{\rho}\rho\cos(\hat{\theta}-\theta)}\right)\,d\hat{\rho}\,d\hat{\theta}$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-\rho^2)\sin\hat{\theta}}{1+\rho^2-2\rho\cos(\hat{\theta}-\theta)}\,d\hat{\theta}.$$

Solution (2 of 2)



Poisson's Equation in Half-Space

Consider Poisson's equation defined on a solid region D in three-dimensional space

$$\begin{aligned} \Delta u &= \phi \text{ for } (x, y, z) \in D \\ u &= f \text{ for } (x, y, z) \in \partial D. \end{aligned}$$

Using the Divergence Theorem,

$$u(x, y, z) = \iiint_D G\phi \, dV + \iint_{\partial D} f G_{\mathbf{N}} \, dS.$$

The Green's function G in many geometries can be found using the method of images.

Poisson's Equation in Half-Space

- Let $D = \{(x, y, z) \,|\, z \geq 0\}$.
- Let the point $\mathbf{x} = (x, y, z)$ be fixed with $z > 0$ and let point $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ be any other point in $\mathbb{R}^3$.
- Denote the distance from $\mathbf{x}$ to $\boldsymbol{\xi}$ by $\rho = ((x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2)^{1/2}$.

The principal solution U for this geometry must satisfy the partial differential equation $\Delta U = \delta(\rho)$.

Consider the Laplacian in spherical coordinates. U will depend only on distance from point $\mathbf{x}$ and satisfies

$$U_{\rho\rho} + \frac{2}{\rho} U_\rho = \delta(\rho).$$

When $\rho > 0$,

$$U(\rho) = \frac{c_1}{\rho} + c_2.$$

Poisson's Equation in Half-Space

Let $\epsilon > 0$ and define $U_\epsilon(\rho) = c_1/(\rho + \epsilon) + c_2$ and set

$$\iiint_{\mathbb{R}^3} \Delta U_\epsilon \, dV = \iiint_{\mathbb{R}^3} \delta(\rho) \, dV.$$

That is

$$\iiint_{\mathbb{R}^3} \frac{-2c_1\epsilon}{\rho(\rho+\epsilon)^3} \, dV = 1,$$

or in spherical coordinates

$$c_1 \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{\epsilon\rho}{(\rho+\epsilon)^3} \sin\varphi \, d\rho \, d\varphi \, d\theta = -\frac{1}{2} \iff c_1 = -\frac{1}{4\pi}.$$

For convenience the arbitrary constant c_2 will be chosen to be zero.

$$U(\rho) = \lim_{\epsilon \to 0^+} U_\epsilon(\rho) = -\frac{1}{4\pi\rho}$$

Method of Images

Include an additional unit point mass so that $G = 0$ on ∂D. The image point denoted $\mathbf{x}'$ will be the reflection of $\mathbf{x}$ across the plane where $z = 0$, therefore $\mathbf{x}' = (x, y, -z)$. Define $\rho' = ((x - \xi)^2 + (y - \eta)^2 + (z + \zeta)^2)^{1/2}$, and then let

$$G(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{4\pi\rho'} - \frac{1}{4\pi\rho}.$$

Green's Function

$G = 0$ on ∂D since for every point $\mathbf{x}_0 = (x_0, y_0, 0) \in \partial D$

$$G = \frac{1}{4\pi((x_0-x)^2+(y_0-y)^2+z^2)^{1/2}} - \frac{1}{4\pi((x_0-x)^2+(y_0-y)^2+z^2)^{1/2}}$$
$$= 0.$$

The unit outward normal of the Green's function along ∂D is $G_{\mathbf{N}} = -[\frac{\partial G}{\partial \zeta}]_{\zeta=0}$.

$$-\frac{\partial G}{\partial \zeta} = \frac{1}{4\pi(\rho')^2}\frac{\partial \rho'}{\partial \zeta} - \frac{1}{4\pi\rho^2}\frac{\partial \rho}{\partial \zeta} = \frac{z+\zeta}{4\pi(\rho')^3} - \frac{z-\zeta}{4\pi\rho^3}$$
$$\left[-\frac{\partial G}{\partial \zeta}\right]_{\zeta=0} = \frac{z}{2\pi\rho^3}.$$

Result

The solution to Poisson's equation where D is the half-space with $z \geq 0$ can be expressed as the integral,

$$u(x, y, z) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \left(\frac{1}{\rho'} - \frac{1}{\rho} \right) \phi \, dV + \frac{z}{2\pi} \iint_{\mathbb{R}^2} \left(\frac{1}{\rho^3} \right) f \, dS.$$

If $\phi = 0$ then the solution to Laplace's equation on the half-space is

$$u(x, y, z) = \frac{z}{2\pi} \iint_{\mathbb{R}^2} \left(\frac{1}{\rho^3} \right) f \, dS.$$

Solving Helmholtz's Equation

Consider the Helmholtz equation in a domain $D \subset \mathbb{R}^2$ with boundary ∂D,

$$\begin{aligned} L[u] = \Delta u + k^2 u &= \phi \text{ for } (x, y) \in D \\ u &= f \text{ for } (x, y) \in \partial D. \end{aligned}$$

We must find the principal solution $U(x, y; \xi, \eta)$ to the differential equation

$$\Delta U + k^2 U = \delta(\xi - x, \eta - y).$$

Polar Coordinates

If a unit point mass is located at (x, y) then the function U depends again only on $r = ((\xi - x)^2 + (\eta - y)^2)^{1/2}$ and hence the PDE can be re-written as

$$\frac{1}{r}(rU_r)_r + k^2 U = \delta(r)$$
$$r(rU_r)_r + k^2 r^2 U = r^2 \delta(r)$$
$$r^2 U_{rr} + rU_r + k^2 r^2 U = r^2 \delta(r).$$

This is the parametric form of Bessel's equation of order zero with linearly independent solutions to the homogeneous form, $J_0(kr)$ and $Y_0(kr)$.

Dirac Delta Behavior

Function $J_0(kr)$ is finite at $r = 0$; however, $Y_0(kr)$ does grow unbounded as $r \to 0^+$.

Let D_ϵ be a disk of radius $\epsilon > 0$ centered at (x, y). Integrate both sides of the equation $\delta(r) = \Delta U + k^2 U$ over the disk.

$$\iint_{D_\epsilon} \delta(r)\, dA = c_1 \iint_{D_\epsilon} (\Delta Y_0(kr) + k^2 Y_0(kr))\, dA$$

$$1 = 2\pi c_1 \left(\int_0^\epsilon r(\Delta Y_0(kr))\, dr + k^2 \int_0^\epsilon r Y_0(kr)\, dr \right)$$

For small $r > 0$,

$$Y_0(kr) \approx \frac{2}{\pi} \ln r.$$

The second integral vanishes as $\epsilon \to 0^+$ since

$$\int_0^\epsilon r Y_0(kr)\, dr \approx \frac{2}{\pi} \int_0^\epsilon r \ln r\, dr = \frac{1}{\pi} \left(\epsilon^2 \ln \epsilon - \frac{1}{2}\epsilon^2 \right) \to 0$$

as $\epsilon \to 0^+$. We can show $U(r) = \frac{1}{4} Y_0(kr)$.

Green's Function for Helmholtz Operator

The Green's function for the Helmholtz operator can be written as the sum of the principal solution U and a function $g(x, y; \xi, \eta)$ such that

$$L^*[g] = g_{\xi\xi} + g_{\eta\eta} + k^2 g = 0.$$

Since $G = 0$ on ∂R then $g = -\frac{1}{4} Y_0(kr)$ on ∂R.

Let D be a disk centered at the origin with radius R. Let $\xi = \hat{\rho}\cos\hat{\theta}$ and $\eta = \hat{\rho}\sin\hat{\theta}$, then the boundary value problem can be written as

$$g_{\hat{\rho}\hat{\rho}} + \frac{1}{\hat{\rho}} g_{\hat{\rho}} + \frac{1}{\hat{\rho}^2} g_{\hat{\theta}\hat{\theta}} + k^2 g = 0$$

$$g(R, \hat{\theta}) = -\frac{1}{4} Y_0\left(k\sqrt{R^2 + \rho^2 - 2R\rho\cos(\hat{\theta} - \theta)}\right).$$

Product Solution

Assume a product solution of the form $g(\hat{\rho}, \hat{\theta}) = \mathrm{P}(\hat{\rho})\Theta(\hat{\theta})$ then

$$\frac{\hat{\rho}^2\mathrm{P}''(\hat{\rho}) + \hat{\rho}\mathrm{P}'(\hat{\rho}) + (k\hat{\rho})^2\mathrm{P}(\hat{\rho})}{\mathrm{P}(\hat{\rho})} = \frac{-\Theta''(\hat{\theta})}{\Theta(\hat{\theta})} = n^2,$$

where n is a nonnegative integer since the $\hat{\theta}$-dependent factor of the solution must be 2π-periodic.

$$\hat{\rho}^2\mathrm{P}''(\hat{\rho}) + \hat{\rho}\mathrm{P}'(\hat{\rho}) + [(k\hat{\rho})^2 - n^2]\mathrm{P}(\hat{\rho}) = 0$$

This is the Bessel equation of order n with a solution defined on the disk of radius R of the form

$$\mathrm{P}_n(\hat{\rho}) = J_n(k\hat{\rho}) \text{ for } n = 0, 1, \ldots.$$

Product Solution

$$\Theta_n(\hat{\theta}) = a_n \cos(n\hat{\theta}) + b_n \sin(n\hat{\theta}).$$

Consequently the term g can be expressed as

$$g(\rho, \theta; \hat{\rho}, \hat{\theta}) = \frac{a_0}{2} J_0(k\hat{\rho}) + \sum_{n=1}^{\infty} \left[J_n(k\hat{\rho})(a_n \cos(n\hat{\theta}) + b_n \sin(n\hat{\theta})) \right].$$

The coefficients a_0, a_n, and b_n can be found using the Euler-Fourier formulas.

Example

Suppose a circular membrane of radius $R = 1$ is centered at the origin. The membrane is subject to a loading force and boundary conditions as described by the following boundary value problem.

$$\Delta u + u = \rho(1 - \rho)\sin\theta \text{ for } 0 \leq \rho < 1 \text{ and } 0 \leq \theta < 2\pi$$
$$u(1, \theta) = 0 \text{ for } 0 \leq \theta < 2\pi.$$

Find the displacement $u(\rho, \theta)$ of the membrane.

Solution (1 of 2)

The principal solution is

$$U(\rho,\theta;\hat{\rho},\hat{\theta}) = \frac{1}{4} Y_0(\sqrt{\hat{\rho}^2 + \rho^2 - 2\hat{\rho}\rho\cos(\hat{\theta}-\theta)}).$$

The function

$$g(\rho,\theta;\hat{\rho},\hat{\theta}) = \frac{a_0}{2} J_0(\hat{\rho}) + \sum_{n=1}^{\infty} \left[J_n(\hat{\rho})(a_n \cos(n\hat{\theta}) + b_n \sin(n\hat{\theta})) \right].$$

The Green's function is

$$G(\rho,\theta;\hat{\rho},\hat{\theta}) = U(\rho,\theta;\hat{\rho},\hat{\theta}) + g(\rho,\theta;\hat{\rho},\hat{\theta}).$$

The solution to the boundary value problem is

$$u(\rho,\theta) = \int_0^{2\pi} \int_0^1 \left(U(\rho,\theta;\hat{\rho},\hat{\theta}) + g(\rho,\theta;\hat{\rho},\hat{\theta}) \right) (1-\hat{\rho}) \sin\hat{\theta}\,\hat{\rho}\, d\hat{\rho}\, d\hat{\theta}.$$

Solution (2 of 2)