Solving Hyperbolic PDEs Using Green's Functions

Partial Differential Equations

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Objectives

In this lesson we will:

- develop Green's functions for hyperbolic partial differential equations, and
- use Green's functions for solving the nonhomogeneous wave equation.

d'Alembert's Solution

Consider the initial value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0 \\ u(x,0) &= f(x) \text{ for } -\infty < x < \infty \\ u_t(x,0) &= g(x) \text{ for } -\infty < x < \infty, \end{aligned}$$

whose solution can be expressed as

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi.$$

Green's Function

$$\frac{1}{2}[f(x-ct)+f(x+ct)] = \frac{1}{2}\int_{-\infty}^{\infty} f(\xi)[\delta(\xi-(x-ct))+\delta(\xi-(x+ct))]\,d\xi$$

The Heaviside function allows the second term of d'Alembert's solution (the "struck string") to be written in the form of an integral over all the real numbers.

$$\frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi = \frac{1}{2c}\int_{-\infty}^{\infty} g(\xi)[H(\xi-(x-ct))-H(\xi-(x+ct))]\,d\xi$$

Define the function

$$G(x,t;\xi,\tau) = \frac{1}{2c}H(c(t-\tau)-|x-\xi|)$$

then d'Alembert's solution can be written as

$$u(x,t) = \int_{-\infty}^{\infty} [f(\xi)G_\tau(x,t;\xi,0)-g(\xi)G(x,t;\xi,0)]\,d\xi.$$

Nonhomogeneous Wave Equation

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= F(x,t) \text{ for } -\infty < x < \infty \text{ and } t > 0 \\
u(x,0) &= f(x) \text{ for } -\infty < x < \infty \\
u_t(x,0) &= g(x) \text{ for } -\infty < x < \infty
\end{aligned}
$$

Replace function v in the adjoint equation with the Green's function.

$$
\iint_R L[u]G\,dA = \oint_{\partial R} [c^2(uG_\xi - u_\xi G)\mathbf{i} + (u_\tau G - uG_\tau)\mathbf{j}] \cdot \mathbf{N}\,ds + \iint_R uL^*[G]\,dA.
$$

The solution can be written as

$$
\begin{aligned}
u(x,t) = &\int_{-\infty}^{\infty} [f(\xi)G_\tau(x,t;\xi,0) - g(\xi)G(x,t;\xi,0)]\,d\xi \\
&+ \int_0^t \int_{-\infty}^{\infty} F(\xi,\tau)G(x,t;\xi,\tau)\,d\xi\,d\tau.
\end{aligned}
$$

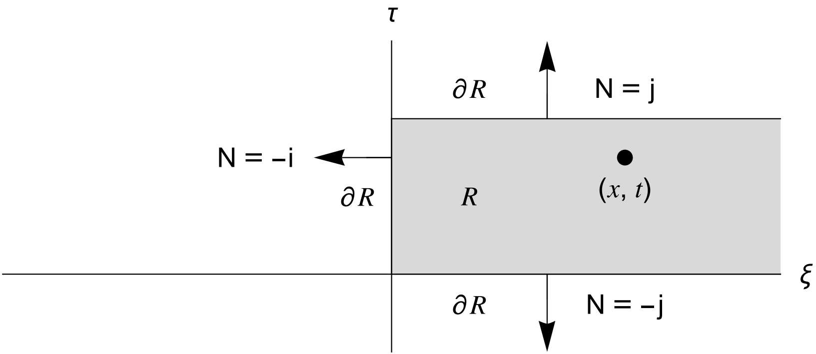
Semi-Infinite Wave Equation

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= F(x,t) \text{ for } x > 0 \text{ and } t > 0 \\
u(x,0) &= f(x) \text{ for } x > 0 \\
u_t(x,0) &= g(x) \text{ for } x > 0 \\
u(0,t) &= h(t) \text{ for } t > 0.
\end{aligned}
$$

The Green's function for the infinite string placed a source at x. For the semi-infinite wave equation suppose in addition to the source at x a source opposite in sign is placed at $-x$. Define the function

$$G(x,t;\xi,\tau) = \frac{H(c(t-\tau) - |x-\xi|) - H(c(t-\tau) - |x+\xi|)}{2c}.$$

Domain



$$
\begin{aligned}
u(x,t) = &\int_0^t \int_0^\infty F(\xi,\tau) G(x,t;\xi,\tau)\, d\xi\, d\tau \\
&+ c^2 \int_0^t h(\tau) G_\xi(x,t;0,\tau)\, d\tau \\
&+ \int_0^\infty \left(f(\xi) G_\tau(x,t;\xi,0) - g(\xi) G(x,t;\xi,0)\right)\, d\xi.
\end{aligned}
$$

Bounded String

Consider the forced wave equation on the finite domain $[0, L]$.

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= F(x,t) \text{ for } 0 < x < L \text{ and } t > 0 \\ \alpha_1 u(0,t) + \beta_1 u_x(0,t) &= 0 \text{ for } 0 < t \\ \alpha_2 u(L,t) + \beta_2 u_x(L,t) &= 0 \text{ for } 0 < t \\ u(x,0) &= 0 \text{ for } 0 < x < L \\ u_t(x,0) &= 0 \text{ for } 0 < x < L. \end{aligned}$$

Eigenfunction Expansion

Suppose the solution can be written in the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\phi_n(x),$$

where $\phi_n(x)$ is the nth orthonormal eigenfunction of the Sturm-Liouville boundary value problem,

$$\begin{aligned} \phi''(x) + \lambda^2\phi(x) &= 0 \text{ for } 0 < x < L \\ \alpha_1\phi(0) + \beta_1\phi'(0) &= 0 \\ \alpha_2\phi(L) + \beta_2\phi'(L) &= 0. \end{aligned}$$

Eigenfunction Expansion

Substitute this solution into the differential equation, use the eigenfunctions and eigenvalues, and integrate over the interval $0 \leq x \leq L$ to produce

$$\sum_{n=1}^{\infty} T_n''(t)\phi_n(x) - c^2 \sum_{n=1}^{\infty} T_n(t)\phi_n''(x) = F(x,t)$$

$$\int_0^L \sum_{n=1}^{\infty} \left(T_n''(t) + c^2\lambda_n^2 T_n(t) \right) \phi_n(x)\phi_m(x)\,dx = \int_0^L F(x,t)\phi_m(x)\,dx$$

$$T_m''(t) + c^2\lambda_m^2 T_m(t) = F_m(t).$$

This time dependent function must satisfy the initial conditions $T_m(0) = T_m'(0) = 0$.

Using the method of variation of parameters the appropriate solution is

$$T_m(t) = \int_0^t \frac{F_m(\tau)\sin(c\lambda_m(t-\tau))}{c\lambda_m}\,d\tau.$$

Substituting this into the series solution reveals

$$u(x,t) = \int_0^t \int_0^L F(\xi,\tau) \left(\sum_{n=1}^{\infty} \frac{\sin(c\lambda_n(t-\tau))\phi_n(\xi)\phi_n(x)}{c\lambda_n} \right) d\xi\,d\tau.$$

Green's Function

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= F(x,t) \text{ for } 0 < x < L \text{ and } t > 0 \\
u(0,t) &= 0 \text{ for } t > 0 \\
u(L,t) &= 0 \text{ for } t > 0 \\
u(x,0) &= f(x) \text{ for } 0 < x < L \\
u_t(x,0) &= g(x) \text{ for } 0 < x < L,
\end{aligned}
$$

The Green's function is

$$
G(x,t;\xi,\tau) = \frac{2}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi \xi}{L}\right) \frac{L}{n\pi c} \sin\left(\frac{n\pi c(t-\tau)}{L}\right).
$$

The solution can be written as

$$
\begin{aligned}
u(x,t) = &\int_0^t \int_0^L G(x,t;\xi,\tau) F(\xi,\tau)\, d\xi\, d\tau \\
&+ \int_0^L G_\tau(x,t;\xi,0) f(\xi)\, d\xi + \int_0^L G(x,t;\xi,0) g(\xi)\, d\xi.
\end{aligned}
$$

Nonhomogeneous Boundary Conditions

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= F(x,t) \text{ for } 0 < x < L \text{ and } t > 0 \\
u(0,t) &= \alpha(t) \text{ for } t > 0 \\
u(L,t) &= \beta(t) \text{ for } t > 0 \\
u(x,0) &= f(x) \text{ for } 0 < x < L \\
u_t(x,0) &= g(x) \text{ for } 0 < x < L.
\end{aligned}
$$

Reference Solution

A reference solution is $r(x,t) = (\beta(t) - \alpha(t))x/L + \alpha(t)$. If $u(x,t) = r(x,t) + v(x,t)$ then the unknown function $v(x,t)$ satisfies the initial boundary value problem

$$
\begin{aligned}
v_{tt} - c^2 v_{xx} &= F(x,t) + (\alpha''(t) - \beta''(t))\frac{x}{L} - \alpha''(t) \text{ for } 0 < x < L \text{ and } t > 0 \\
v(0,t) &= 0 \text{ for } t > 0 \\
v(L,t) &= 0 \text{ for } t > 0 \\
v(x,0) &= f(x) + (\alpha(0) - \beta(0))\frac{x}{L} - \alpha(0) \text{ for } 0 < x < L \\
v_t(x,0) &= g(x) + (\alpha'(0) - \beta'(0))\frac{x}{L} - \alpha'(0) \text{ for } 0 < x < L,
\end{aligned}
$$

which is of the form shown previously and thus $v(x,t)$ can calculated using the previous formula.