Solving Parabolic PDEs Using Green's Functions

Partial Differential Equations

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Objectives

In this lesson we will:

- develop Green's functions for some parabolic differential equations, and
- use Green's functions to solve examples of the heat or diffusion equation.

One-Dimensional Setting

$$
\begin{aligned}
u_t - \kappa u_{xx} &= Q(x,t) \text{ for } 0 < x < M \text{ and } t > 0 \\
u(0,t) &= \alpha(t) \text{ for } t > 0 \\
u(M,t) &= \beta(t) \text{ for } t > 0 \\
u(x,0) &= f(x) \text{ for } 0 < x < M
\end{aligned}
$$

Eigenfunction Expansion

Write the solution as an infinite sum involving the eigenfunctions of the Dirichlet-bounded heat equation in one spatial dimension.

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\phi_n(x),$$

where $\phi_n(x) = \sin\frac{n\pi x}{M}$. Differentiate the solution with respect to t and solve for $T_n'(t)$.

$$\sum_{n=1}^{\infty} T_n'(t)\phi_n(x) = u_t = \kappa u_{xx} + Q(x,t)$$

Multiply both sides by $\phi_m(x)$ and integrate over the interval $[0, M]$.

$$\int_0^M \sum_{n=1}^{\infty} T_n'(t)\phi_n(x)\phi_m(x)\,dx = \int_0^M (\kappa u_{xx} + Q(x,t))\phi_m(x)\,dx$$

Assuming the order of integrations and summation can be interchanged, the orthogonality of the eigenfunctions implies

$$T_m'(t) = \frac{\kappa\int_0^M u_{xx}\phi_m(x)\,dx}{\|\phi_m\|^2} + \frac{\int_0^M Q(x,t)\phi_m(x)\,dx}{\|\phi_m\|^2}$$

Eigenfunction Expansion

Define $L[u] = u_{xx}$. By integration by parts or Green's Formula

$$\int_0^M (uL[\phi_m] - \phi_m L[u])\, dx = [u\phi'_m - u_x\phi_m]_{x=0}^{x=M}$$

$$\begin{aligned}\int_0^M (-\lambda_m u - u_{xx})\phi_m\, dx &= u(M,t)\phi'_m(M) - u_x(M,t)\phi_m(M) \\ &\quad - u(0,t)\phi'_m(0) + u_x(0,t)\phi_m(0) \\ &= \beta(t)\phi'_m(M) - \alpha(t)\phi'_m(0).\end{aligned}$$

Replace u by the series formulation and use the orthogonality of the eigenfunctions.

$$\int_0^M -\lambda_m u\phi_m\, dx = \int_0^M -\lambda_m \left(\sum_{n=1}^{\infty} T_n(t)\phi_n(x)\right)\phi_m(x)\, dx = -\lambda_m T_m(t)\|\phi_m\|^2$$

Thus the necessary expression for the integral is

$$\int_0^M u_{xx}\phi_m\, dx = -\lambda_m T_m(t)\|\phi_m\|^2 - \beta(t)\phi'_m(M) + \alpha(t)\phi'_m(0).$$

Eigenfunction Expansion

$$T_m'(t) = \frac{\kappa\left(-\lambda_m T_m(t)\|\phi_m\|^2 - \beta(t)\phi_m'(M) + \alpha(t)\phi_m'(0)\right)}{\|\phi_m\|^2} + \frac{\int_0^M Q(x,t)\phi_m(x)\,dx}{\|\phi_m\|^2}.$$

Let the last expression be denoted $Q_m(t)$ and a nonhomogeneous, first-order ordinary differential equation for T_m arises.

$$T_m'(t) + \kappa\lambda_m T_m(t) = Q_m(t) + \kappa\frac{\alpha(t)\phi_m'(0) - \beta(t)\phi_m'(M)}{\|\phi_m\|^2}$$

Solving the differential equation above yields

$$T_m(t) = \int_0^t e^{-\kappa\lambda_m(t-\tau)}\left[Q_m(\tau) + \kappa\frac{\alpha(\tau)\phi_m'(0) - \beta(\tau)\phi_m'(M)}{\|\phi_m\|^2}\right]\,d\tau + T_m(0)e^{-\kappa\lambda_m t}.$$

Eigenfunction Expansion

$$u(x,t) = \sum_{n=1}^{\infty} T_n(0) e^{-\kappa\lambda_n t}\phi_n(x)$$
$$+ \sum_{n=1}^{\infty} \left(\int_0^t e^{-\kappa\lambda_n(t-\tau)} \left[Q_n(\tau) + \kappa \frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(M)}{\|\phi_n\|^2} \right] d\tau \right) \phi_n(x)$$

Note that when $t = 0$,

$$\sum_{n=1}^{\infty} T_n(0)\phi_n(x) = u(x,0) = f(x)$$

$$T_n(0) = \frac{\int_0^M f(x)\phi_n(x)\,dx}{\|\phi_n\|^2}.$$

Eigenfunction Expansion

$$
\begin{aligned}
u(x,t) = &\int_0^M \left(\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\|\phi_n\|^2} e^{-\kappa\lambda_n t} \right) f(\xi)\, d\xi \\
&+ \int_0^t \int_0^M \left(\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\|\phi_n\|^2} e^{-\kappa\lambda_n (t-\tau)} \right) Q(\xi,\tau)\, d\xi\, d\tau \\
&+ \kappa \int_0^t \left(\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n'(0)}{\|\phi_n\|^2} e^{-\kappa\lambda_n (t-\tau)} \right) \alpha(\tau)\, d\tau \\
&- \kappa \int_0^t \left(\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n'(M)}{\|\phi_n\|^2} e^{-\kappa\lambda_n (t-\tau)} \right) \beta(\tau)\, d\tau.
\end{aligned}
$$

Green's Function

Define the Green's function is

$$G(x,t;\xi,\tau) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\|\phi_n\|^2} e^{-\kappa\lambda_n(t-\tau)}.$$

The solution to the IBVP can be expressed as

$$u(x,t) = \int_0^M G(x,t;\xi,0)f(\xi)\,d\xi + \int_0^t \int_0^M G(x,t;\xi,\tau)Q(\xi,\tau)\,d\xi\,d\tau$$
$$+\kappa \int_0^t \left[G_\xi(x,t;0,\tau)\alpha(\tau) - G_\xi(x,t;M,\tau)\beta(\tau)\right]\,d\tau.$$

The first integral above handles the initial condition, the second addresses the source term, and the third was previously referred to as the surface terms.

Infinite Domain

The principal solution term of the Green's function for the unbounded one-dimensional heat equation must satisfy the differential equation

$$L^*[U] = -U_\tau - \kappa U_{\xi\xi} = \delta(\xi - x, \tau - t) \text{ for } -\infty < \xi < \infty \text{ and } \tau > 0.$$

Apply the Fourier transform.

$$\begin{aligned} \mathcal{F}[-U_\tau - \kappa U_{\xi\xi}] &= \mathcal{F}[\delta(\xi - x, \tau - t)] \\ -\hat{U}_\tau + \omega^2 \kappa \hat{U} &= \frac{1}{\sqrt{2\pi}} \delta(\tau - t) e^{-i\omega x} \end{aligned}$$

The solution to this first-order ordinary differential equation is

$$\hat{U}(\omega) = \begin{cases} A e^{\omega^2 \kappa \tau} & \text{if } t < \tau, \\ B e^{\omega^2 \kappa \tau} & \text{if } t > \tau. \end{cases}$$

Infinite Domain

Let $\epsilon > 0$ and integrate both sides of the ODE with respect to τ over the interval $[t - \epsilon, t + \epsilon]$.

$$\int_{t-\epsilon}^{t+\epsilon} (-\hat{U}_\tau + \omega^2 \kappa \hat{U})\, d\tau = \int_{t-\epsilon}^{t+\epsilon} \frac{1}{\sqrt{2\pi}} \delta(\tau - t) e^{-i\omega x}\, d\tau$$

$$-Ae^{\omega^2\kappa(t+\epsilon)} + Be^{\omega^2\kappa(t-\epsilon)} + \int_{t-\epsilon}^{t+\epsilon} \omega^2 \kappa \hat{U}\, d\tau = \frac{1}{\sqrt{2\pi}} e^{-i\omega x}$$

Taking the limit of both sides of the last equation as $\epsilon \to 0$ produces the equation,

$$(B - A)e^{\omega^2 \kappa t} = \frac{1}{\sqrt{2\pi}} e^{-i\omega x}.$$

Infinite Domain

If $\tau > t$ the differential equation for the principal solution is $L^*[U] = -U_\tau - \kappa U_{\xi\xi} = 0$ which has solution $U = 0$. This implies that for $\tau > t$, the Fourier transform $\hat{U} = 0$ and $A = 0$. Consequently

$$B = \frac{1}{\sqrt{2\pi}} e^{-(i\omega x + \omega^2 \kappa t)} \text{ and } \hat{U}(\omega) = \frac{H(t-\tau)}{\sqrt{2\pi}} e^{-i\omega x - \omega^2 \kappa (t-\tau)}.$$

Apply the inverse Fourier transform to $\hat{U}$ to determine the principal solution portion of Green's function.

$$U(x, t; \xi, \tau) = \frac{H(t-\tau)}{\sqrt{4\pi\kappa(t-\tau)}} e^{-\frac{(\xi - x)^2}{4\kappa(t-\tau)}}.$$

Since the other component of the Green's function must satisfy $L^*[g] = 0$ subject to the boundary conditions at infinity then $g(x, t; \xi, \tau) = 0$. Thus the Green's function $G(x, t; \xi, \tau) = U(x, t; \xi, \tau)$.

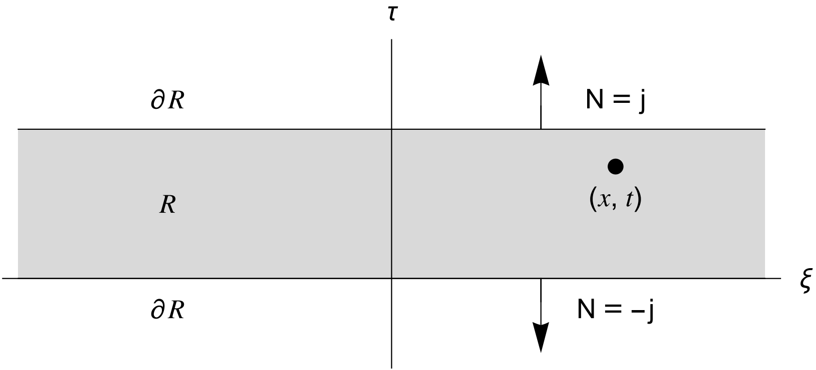
Initial Value Problem

$$L[u] = u_t - \kappa u_{xx} = Q(x,t) \text{ for } -\infty < x < \infty \text{ and } t > 0$$
$$u(x,0) = f(x) \text{ for } -\infty < x < \infty.$$

The formal adjoint equation is defined over a region R. Replacing v with the Green's function results in the equation,

$$\iint_R L[u]G\,dA = \oint_{\partial R} [(-\kappa u_\xi G + \kappa u G_\xi)\mathbf{i} + uG\mathbf{j}] \cdot \mathbf{N}\,ds + \iint_R u L^*[G]\,dA.$$

Region R



$$u(x,t) = \int_{-\infty}^{\infty} f(\xi) G(x,t;\xi,0)\, d\xi + \int_0^t \int_{-\infty}^{\infty} Q(\xi,\tau) G(x,t;\xi,\tau)\, d\xi\, d\tau.$$

Semi-Infinite Domain

$$\begin{aligned} L[u] = u_t - \kappa u_{xx} &= Q(x,t) \text{ for } 0 < x < \infty \text{ and } t > 0 \\ u(0,t) &= h(t) \text{ for } t > 0 \\ u(x,0) &= f(x) \text{ for } 0 < x < \infty. \end{aligned}$$

The Green's function for the semi-infinite domain is

$$G(x,t;\xi,\tau) = U(x,t;\xi,\tau) - U(-x,t;\xi,\tau).$$

The solution can be expressed as

$$\begin{aligned} u(x,t) = &\int_0^t \kappa h(\tau) G_\xi(x,t;0,\tau)\,d\tau + \int_0^\infty f(\xi) G(x,t;\xi,0)\,d\xi \\ &+ \int_0^t \int_0^\infty Q(\xi,\tau) G(x,t;\xi,\tau)\,d\xi\,d\tau. \end{aligned}$$

Example

Consider the initial, boundary value problem,

$$\begin{aligned} u_t - u_{xx} &= 1 + \sin(x\,t) \text{ for } 0 < x < \infty \text{ and } t > 0 \\ u(0,t) &= e^{-t} \text{ for } t > 0 \\ u(x,0) &= \cos^2 x \text{ for } 0 < x < \infty. \end{aligned}$$

Solution (1 of 2)

The solution may be formally written as the sum of the integrals,

$$u(x,t) = \int_0^t e^{-\tau} G_\xi(x,t;0,\tau)\,d\tau + \int_0^\infty (\cos^2\xi) G(x,t;\xi,0)\,d\xi$$
$$+ \int_0^t \int_0^\infty (1+\sin(\xi\tau)) G(x,t;\xi,\tau)\,d\xi\,d\tau.$$

Solution (2 of 2)