Green's Functions for Sturm-Liouville Boundary Value Problems

Partial Differential Equations

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Objectives

In this lesson we will:

- review the Sturm-Liouville boundary value problem,
- develop a Green's function for the Sturm-Liouville BVP,
- present some of the properties of the Green's function for the Sturm-Liouville BPV, and
- use the Green's function to solve a Sturm-Liouville BVP.

Sturm-Liouville Boundary Value Problem

$$L[u](x) = [p(x)u'(x)]' + q(x)u(x) = f(x) \text{ for } a < x < b$$
$$\alpha_1 u(a) + \beta_1 u'(a) = 0$$
$$\alpha_2 u(b) + \beta_2 u'(b) = 0,$$

where $\alpha_1^2 + \beta_1^2 > 0$ and $\alpha_2^2 + \beta_2^2 > 0$.

Particular Solution

Assume $u_1(x)$ and $u_2(x)$ are linearly independent solutions to the homogeneous form of the ordinary differential equation.

Define the Wronskian

$$W[u_1, u_2](x) = u_1(x)u_2'(x) - u_1'(x)u_2(x)$$

A particular solution to the nonhomogeneous ODE is

$$u_p(x) = -u_1(x)\int_{\xi_1}^{x} \frac{f(y)u_2(y)}{p(y)W[u_1, u_2](y)}\,dy + u_2(x)\int_{\xi_2}^{x} \frac{f(y)u_1(y)}{p(y)W[u_1, u_2](y)}\,dy,$$

where ξ_1 and ξ_2 are parameters.

Remark: the denominators of the definite integrals above are constant on interval (a, b) though the value of the constant depends on the choice of homogeneous solutions $u_1(x)$ and $u_2(x)$.

General Solution

The general solution to the nonhomogeneous equation can be written as

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + u_p(x).$$

This general solution can be written in the compact form

$$u(x) = -\int_{z_1}^{x} \frac{f(y)u_1(x)u_2(y)}{p\,W}\,dy + \int_{z_2}^{x} \frac{f(y)u_1(y)u_2(x)}{p\,W}\,dy$$

with the appropriate choice of limits of integration z_1 and z_2. These limits of integration are determined by the boundary conditions.

$$u(x) = \int_{a}^{x} \frac{f(y)u_1(y)u_2(x)}{p\,W}\,dy + \int_{x}^{b} \frac{f(y)u_1(x)u_2(y)}{p\,W}\,dy$$

Green's Function

$$G(x;y) = \begin{cases} \frac{u_1(x)u_2(y)}{p\,W} & \text{if } a \leq x < y \leq b \\ \frac{u_1(y)u_2(x)}{p\,W} & \text{if } a \leq y < x \leq b. \end{cases}$$

$G(x;y)$ permits the solution to the boundary value problem to be expressed concisely as

$$u(x) = \int_a^b G(x;y)f(y)\,dy.$$

Remark: the symbol y is a supplemental variable present in the Green's function which is eliminated from the final solution by the process of integration.

Differential Operator

Define a general second-order linear differential operator as

$$L[u] = \alpha(x)\frac{d^2u}{dx^2} + \beta(x)\frac{du}{dx} + \gamma(x)u.$$

Let $\mathcal{D}(L)$ denote the domain of the differential operator L, that is, the set of functions u for which $L[u]$ is defined and which satisfy the imposed boundary conditions. The **adjoint operator** to differential operator L is denoted L^* and is the operator satisfying the equation:

$$\langle L[u], v\rangle = \langle u, L^*[v]\rangle.$$

The expressions $\langle L[u], v\rangle$ and $\langle u, L^*[v]\rangle$ denote an inner product of functions on interval $[a, b]$.

Finding the Adjoint Operator

Suppose $u(x)$ and $v(x)$ are twice differentiable functions defined on $[a, b]$, and

$$\left\langle \alpha(x)\frac{d^2u}{dx^2} + \beta(x)\frac{du}{dx} + \gamma(x)u, v \right\rangle = \langle u, L^*[v]\rangle$$

$$\int_a^b \alpha(x)\frac{d^2u}{dx^2}v(x) + \beta(x)\frac{du}{dx}v(x) + \gamma(x)u(x)v(x)\,dx = \int_a^b u(x)L^*[v](x)\,dx.$$

Using integration by parts yields

$$\begin{aligned}\langle L[u], v\rangle = {} & \left[u'(x)\alpha(x)v(x) - u(x)[\alpha(x)v(x)]' + u(x)\beta(x)v(x)\right]_{x=a}^{x=b} \\ & + \int_a^b \left([\alpha(x)v(x)]'' - [\beta(x)v(x)]' + \gamma(x)v(x)\right)u(x)\,dx.\end{aligned}$$

Define the differential operator L^* as

$$L^*[v](x) = \frac{d^2}{dx^2}\left[\alpha(x)\,v(x)\right] - \frac{d}{dx}\left[\beta(x)\,v(x)\right] + \gamma(x)\,v(x).$$

Remarks

$$L^*[v](x) = \frac{d^2}{dx^2}\left[\alpha(x)\, v(x)\right] - \frac{d}{dx}\left[\beta(x)\, v(x)\right] + \gamma(x)\, v(x).$$

L^* is known as the **formal adjoint** to L.

The formal adjoint combined with the boundary conditions which function $v(x)$ must satisfy at $x = a$ and $x = b$ constitute the adjoint. The expression,

$$\left[u'(x)\alpha(x)v(x) - u(x)[\alpha(x)v(x)]' + u(x)\beta(x)v(x)\right]_{x=a}^{x=b}$$

will be known as the **surface terms**

Generalization of Green's Formula

$$\langle L[u], v\rangle - \langle u, L^*[v]\rangle = \left[u'(x)\alpha(x)v(x) - u(x)[\alpha(x)v(x)]' + u(x)\beta(x)v(x)\right]_{x=a}^{x=b}.$$

If $L = L^*$ then the differential operator L is described as **formally self-adjoint**.

If in addition the functions upon which these transformations operate satisfy the same boundary conditions, then L is **self-adjoint**.

Properties

$$L[u](x) = \left[p(x)u'(x)\right]' + q(x)u(x) = f(x) \text{ for } a < x < b$$
$$\alpha_1 u(a) + \beta_1 u'(a) = 0$$
$$\alpha_2 u(b) + \beta_2 u'(b) = 0,$$

where $\alpha_1^2 + \beta_1^2 > 0$ and $\alpha_2^2 + \beta_2^2 > 0$.

Theorem

A Green's function

$$G(x; y) = \begin{cases} \frac{u_1(x)u_2(y)}{p\,W} & \text{if } a \le x < y \le b \\ \frac{u_1(y)u_2(x)}{p\,W} & \text{if } a \le y < x \le b. \end{cases}$$

for the Sturm-Liouville boundary value problem has the following properties.

(i) $G(x; y)$ *satisfies the homogeneous ordinary differential equation* $L[G](x) = 0$ *for* $x \neq y$.

(ii) $G(x; y)$ *satisfies the boundary conditions of the Sturm-Liouville BVP.*

(iii) $G(x; y) = G(y; x)$ *for all* $a \le x, y \le b$.

(iv) $G(x; y)$ *is continuous for* $(x, y) \in (a, b) \times (a, b)$.

(v) $G_x(x; y)$ *is continuous for* $x \neq y$ *and has a jump discontinuity across the line* $x = y$,

Crucial Property of Green's Function

Green's function satisfies the homogeneous differential equation $L[G](x) = 0$ when $x \neq y$ but $G_x(x; y)$ does not exist when $x = y$.

For $(x, y) \in (a, b) \times (a, b)$, the Green's function satisfies the nonhomogeneous differential equation

$$L[G](x) = \delta(y - x),$$

where $\delta(y - x)$ is the Dirac delta function.

Example

Use the Green's function approach to find the solution to the boundary value problem:

$$u''(x) + 16u(x) = (x+1)^2 \text{ for } 0 < x < 1$$
$$u(0) = u'(1) = 0.$$

Solution (1 of 6)

Find a solution to

$$\begin{aligned} G_{xx}(x;y) + 16G(x;y) &= 0 \text{ for } 0 < x < 1,\ x \neq y \\ G(0;y) = G_x(1;y) &= 0. \end{aligned}$$

The general solution is $G(x;y) = c_1 \sin(4x) + c_2 \cos(4x)$ and thus the Green's function may be expressed as

$$G(x;y) = \begin{cases} a_1 \sin(4x) + a_2 \cos(4x) & \text{if } 0 \leq x < y \leq 1 \\ b_1 \sin(4x) + b_2 \cos(4x) & \text{if } 0 \leq y < x \leq 1, \end{cases}$$

where a_1, a_2, b_1, and b_2 are functions of the parameter y.

Solution (2 of 6)

The coefficients a_1, a_2, b_1, and b_2 must be chosen so that

- $G(x; y)$ satisfies the same boundary conditions as solution $u(x)$,
- $G(x; y)$ is continuous for $(x, y) \in (0, 1) \times (0, 1)$, and
- the function $G_x(x; y)$ has a jump discontinuity of the form $G_x(y+; y) - G_x(y-; y) = 1$.

The boundary condition $G(0; y) = 0$ forces $a_2 = 0$. The continuity of $G(x; y)$ implies

$$(a_1 - b_1)\sin(4y) + b_2 \cos(4y) = 0.$$

The jump discontinuity in the partial derivative implies

$$(-4a_1 + 4b_1)\cos(4y) - 4b_2 \sin(4y) = 1.$$

Solution (3 of 6)

$$\begin{aligned}(a_1 - b_1)\sin(4y) + b_2\cos(4y) &= 0\\(-4a_1 + 4b_1)\cos(4y) - 4b_2\sin(4y) &= 1\end{aligned}$$

Solving this system of two equations yields

$$\begin{aligned}a_1 - b_1 &= -\frac{1}{4}\cos(4y)\\b_2 &= -\frac{1}{4}\sin(4y).\end{aligned}$$

Solution (4 of 6)

The boundary condition $G_x(1; y) = 0$ implies

$$0 = 4b_1 \cos(4) - 4b_2 \sin(4) = 4b_1 \cos(4) + \sin(4y) \sin(4)$$

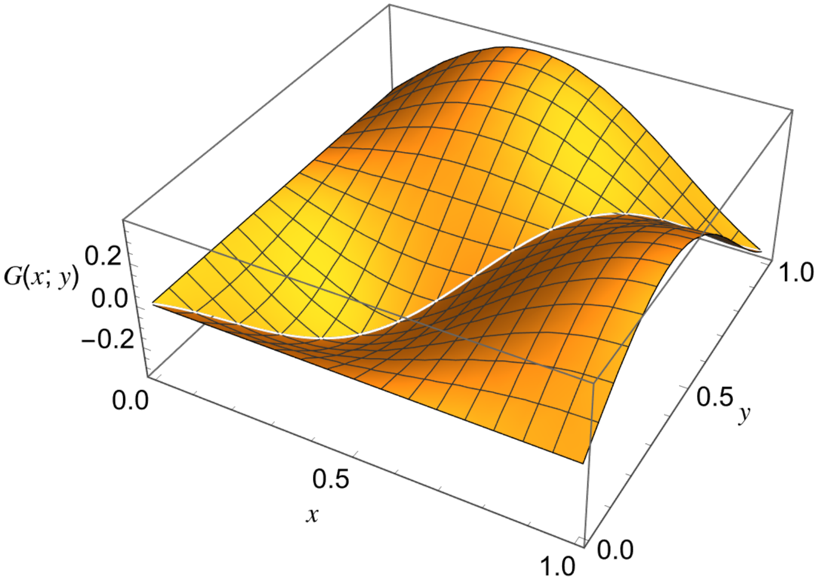
using the value found for b_2 which in turn determines

$$b_1 = \frac{-\sin(4y)\sin(4)}{4\cos(4)} \text{ and } a_1 = \frac{-\cos(4y-4)}{4\cos(4)}.$$

Consequently the Green's function for this boundary value problem takes the form,

$$G(x; y) = -\frac{1}{4\cos(4)} \begin{cases} \sin(4x)\cos(4y-4) & \text{if } 0 \le x < y \le 1 \\ \cos(4x-4)\sin(4y) & \text{if } 0 \le y < x \le 1. \end{cases}$$

Solution (5 of 6)



Solution (6 of 6)

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\begin{aligned}
u(x) &= \int_0^1 G(x;y)f(y)\,dy \\
&= \int_0^x \frac{-\sin(4y)\cos(4x-4)}{4\cos(4)}(y+1)^2\,dy \\
&\quad + \int_x^1 \frac{-\sin(4x)\cos(4y-4)}{4\cos(4)}(y+1)^2\,dy \\
&= \frac{-\cos(4x-4)}{4\cos(4)}\int_0^x (y+1)^2\sin(4y)\,dy \\
&\quad - \frac{\sin(4x)}{4\cos(4)}\int_x^1 (y+1)^2\cos(4y-4)\,dy \\
&= \frac{1}{16}\left((x+1)^2 - \frac{1}{8}\right) - \frac{7\cos(4x-4)}{128\cos(4)} - \frac{\sin(4x)}{16\cos(4)}.
\end{aligned}
$$