Burgers' Equation

Partial Differential Equations

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Objectives

In this lesson we will:

- introduce Burgers' equation,
- examine conservation laws' role in determining solutions,
- develop traveling wave solutions to Burgers' equation,
- introduce the Cole-Hopf transformation,
- introduce the inviscid form of Burgers' equation, and
- recognize conditions forming shocks and rarefactions in solutions to PDEs.

Standard Form of Burgers' Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0,$$

where the diffusion coefficient $\nu > 0$ is a constant.

- Burgers' equation was first proposed to model the motion of turbulent fluid flow.
- The nonlinear term $u\,u_x$ models a convection process.
- The term $\nu\,u_{xx}$ models the effect of viscosity or the diffusion process.
- If $\nu = 0$, the equation is known as the inviscid Burgers' equation.

Conservation Laws

- The solution to Burgers' equation is sometimes not unique or is multi-valued.
- If a quantity must be conserved in Burgers' equation, that quantity can be used to determine which of the multiple values (if any) is the "correct" solution to Burgers' equation.

Integrate the viscous Burgers' equation with respect to x over an arbitrary interval $[a, b]$.

$$\int_a^b u_t\,dx + \int_a^b u\,u_x\,dx - \nu \int_a^b u_{xx}\,dx = 0$$

$$\int_a^b u_t\,dx + \frac{1}{2}(u(b,t))^2 - \frac{1}{2}(u(a,t))^2 - \nu u_x(b,t) + \nu u_x(a,t) = 0$$

By Leibniz's rule,

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \int_a^b u_t\,dx + u(b,t)\frac{db}{dt} - u(a,t)\frac{da}{dt}.$$

Conserved Quantity

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \left(u(b,t)\frac{db}{dt} - \frac{1}{2}(u(b,t))^2 + \nu u_x(b,t)\right) - \left(u(a,t)\frac{da}{dt} - \frac{1}{2}(u(a,t))^2 + \nu u_x(a,t)\right).$$

The rate of change of $\int_a^b u(x,t)\,dx$ is the quantity,

$$\left[u(x,t)\frac{dx}{dt} - \frac{1}{2}(u(x,t))^2 + \nu\, u_x(x,t)\right]_{x=a}^{x=b}.$$

The rate of change of the integral of u over $[a,b]$ is dependent only on the values at the endpoints of the integral (namely a and b). Thus the integral of u is said to be *conserved*.

Traveling Wave Solutions

Set $\xi = x - c\,t$ and substitute $u(x,t) = U(\xi)$ into Burgers' equation.

$$-c\,\frac{dU}{d\xi} + u\,\frac{dU}{d\xi} - \nu\,\frac{d^2U}{d\xi^2} = 0$$

Integrate both sides of the equation with respect to ξ.

$$-c\,U + \frac{1}{2}U^2 - \nu\,\frac{dU}{d\xi} = A,$$

where A is a constant. Therefore,

$$\frac{dU}{d\xi} = \frac{1}{2\nu}(U^2 - 2c\,U - 2A).$$

Remarks

- There is no bounded solution if the quadratic equation $U^2 - 2c\,U - 2A = 0$ has only one real root (other than the constant solution) or two complex roots.
- Assume $U^2 - 2c\,U - 2A = 0$ has two distinct real roots, denoted $U_1 = c - \sqrt{c^2 + 2A}$ and $U_2 = c + \sqrt{c^2 + 2A}$.
- The constant functions $U(\xi) = U_1$ and $U(\xi) = U_2$ are equilibrium solutions.
- If $U(\xi)$ is a solution to the ODE with initial value between U_1 and U_2, then by the uniqueness of solutions to the ordinary differential equation, $U(\xi)$ will be bounded and stay between U_1 and U_2.

Separation of Variables (1 of 2)

Separate the variables and rewrite the ODE as

$$\frac{1}{(U - U_1)(U - U_2)} \frac{dU}{d\xi} = \frac{1}{2\nu}$$

Integrate with respect to ξ.

$$\ln \left| \frac{U - U_1}{U - U_2} \right| = \frac{(U_1 - U_2)}{2\nu} \xi + B.$$

where B is an arbitrary constant of integration. Use the fact that $U_1 < U(\xi) < U_2$,

$$\frac{U - U_1}{U_2 - U} = e^{\alpha\xi + B}.$$

where $\alpha = (U_1 - U_2)/(2\nu) < 0$. Solve this equation for U.

$$U(\xi) = \frac{U_1 + U_2 e^{\alpha\xi + B}}{1 + e^{\alpha\xi + B}}.$$

Separation of Variables (2 of 2)

The solution U satisfies

$$\lim_{\xi \to -\infty} U(\xi) = U_2 \text{ and } \lim_{\xi \to \infty} U(\xi) = U_1.$$

The traveling wave solution for Burgers' equation:

$$u(x,t) = U(x - c\,t) = \frac{U_1 + U_2 e^{\alpha(x - c\,t) + B}}{1 + e^{\alpha(x - c\,t) + B}},$$

Special Case

If $B = 0$, the traveling wave solution becomes

$$u(x,t) = U(x - c\,t) = \frac{U_1 + U_2 e^{\alpha(x - c\,t)}}{1 + e^{\alpha(x - c\,t)}}$$

and the initial wave profile is obtained by setting $t = 0$,

$$u(x,0) = U(x) = \frac{U_1 + U_2 e^{\alpha x}}{1 + e^{\alpha x}}.$$

Only when the initial condition is given by the function above or one of its horizontal translations does this lead to a traveling wave solution.

Properties of Traveling Wave Solutions

1. For any fixed t, a traveling wave solution decreases from U_2 to U_1 as x increases from $-\infty$ to ∞.
2. The traveling waves travel from the left to right with a speed $c = (U_1 + U_2)/2$ since U_1 and U_2 are solutions of the quadratic equation $U^2 - 2c\,U - 2A = (U - U_1)(U - U_2) = 0$.
3. As the viscosity ν approaches 0, for fixed t and x, if $x < c\,t$, the solution $u(x,t) \to U_2$ and if $x > c\,t$, $u(x,t) \to U_1$. In the limit, a shock wave develops.

Arbitrary Initial Conditions

Question: does Burgers' equation have a solution for any given initial condition,

$$u(x,0) = f(x) \text{ for } -\infty < x < \infty.$$

If there exists a solution to the initial value problem, is the solution unique and is there a way to find the solution explicitly?

Cole-Hopf Transformation (1 of 2)

Rewrite Burgers' equation as

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 - \nu\frac{\partial u}{\partial x}\right) = 0 \text{ for } -\infty < x < \infty \text{ and } t > 0.$$

The Cole-Hopf transformation is defined by

$$u = -2\nu\frac{v_x}{v} \iff u = -2\nu\frac{\partial}{\partial x}\left[\ln v\right],$$

where it is implicitly assumed that the unknown function $v(x,t) > 0$.
The chain rule implies

$$\begin{aligned}\frac{\partial u}{\partial t} &= -2\nu\frac{v\,v_{xt} - v_x v_t}{v^2} = -2\nu\frac{v\,v_{tx} - v_t v_x}{v^2} = -2\nu\frac{\partial}{\partial x}\left[\frac{v_t}{v}\right]\\ \frac{\partial u}{\partial x} &= -2\nu\frac{v\,v_{xx} - (v_x)^2}{v^2}.\end{aligned}$$

Cole-Hopf Transformation (2 of 2)

This allows Burgers' equation to be written as

$$-2\nu \frac{\partial}{\partial x}\left[\frac{v_t}{v}\right] + \frac{\partial}{\partial x}\left[2\nu^2 \frac{v_{xx}}{v}\right] = 0,$$

which simplifies to the following:

$$\frac{\partial}{\partial x}\left[\frac{v_t - \nu\, v_{xx}}{v}\right] = 0.$$

Remark: if a function $v(x,t) > 0$ is a solution of the linear, homogeneous heat equation

$$v_t = \nu\, v_{xx},$$

the numerator of the term inside the partial derivative will be zero and thus $u(x,t)$ is a solution to Burgers' equation.

Conversely if $u(x,t)$ is any solution to Burgers' equation, the function

$$v(x,t) = e^{-\frac{1}{2\nu}\int_0^x u(y,t)\,dy}$$

solves the transformed equation.

Example

Function $v(x,t) = A + Be^{-\nu t}\sin x$ is a solution of the heat equation for any constants A and B,.

If A is large enough to guarantee that $v(x,t)$ is positive everywhere,

$$u(x,t) = \frac{-2\nu Be^{-\nu t}\cos x}{A + Be^{-\nu t}\sin x}$$

is a solution to Burgers's equation.

Burgers' Equation IVP

Suppose the initial condition for Burgers' equation is $u(x,0) = f(x)$, then

$$v(x,0) = g(x) = e^{-\frac{1}{2\nu}\int_0^x f(y)\,dy}.$$

and the solution to the heat equation is

$$v(x,t) = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty} g(y)e^{-\frac{(x-y)^2}{4\nu t}}\,dy \text{ for } t > 0.$$

Therefore

$$u(x,t) = \frac{\int_{-\infty}^{\infty}(x-y)e^{-\frac{1}{2\nu}\left(\int_0^y f(s)\,ds + \frac{(x-y)^2}{2t}\right)}\,dy}{t\int_{-\infty}^{\infty} e^{-\frac{1}{2\nu}\left(\int_0^y f(s)\,ds + \frac{(x-y)^2}{2t}\right)}\,dy}.$$

Inviscid Burgers' Equation

If $\nu = 0$, then Burgers' equation becomes

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0,$$

known as the inviscid Burgers' equation.

Use the method of characteristics to solve the inviscid Burgers' equation.

$$\frac{dx}{ds} = u, \quad \frac{dt}{ds} = 1, \text{ and } \frac{du}{ds} = 0.$$

The 3rd characteristic equation indicates that u is constant along characteristics. The 2nd characteristic equation allows the parameter s to be identified with the independent variable t. Hence the 1st characteristic equation may be rewritten as $dx/dt = u$ and thus

$$x = u\,t + \gamma_1$$
$$u = \gamma_2$$

where γ_1 and γ_2 are arbitrary constants which may be determined from initial conditions imposed on the inviscid Burgers' equation.

Inviscid Burgers' Equation IVP

If the initial condition is $u(x, 0) = f(x)$, then along the characteristic through $(x_0, 0)$ it is the case that $\gamma_1 = x_0$ and thus $x_0 = x - u\,t$ on the characteristic. Therefore the solution can be written in implicit form as

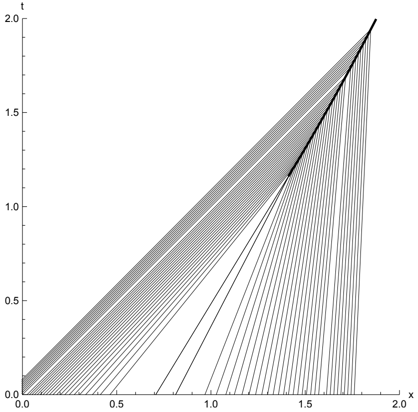
$$u(x, t) = f(x_0) = f(x - u(x, t)t).$$

For example if $u(x, 0) = e^{-x^2}$, the implicit form of the solution is

$$u(x, t) = e^{-(x-u(x,t)t)^2}.$$

Intersection of Characteristics

The characteristics are straight lines with slopes that depend on u, so unless u is a constant, these characteristics may intersect for some $t > 0$.



The thick curve is the locus of points where characteristics intersect and is called a shock.

Shocks and Rarefactions

- If the t-axis is oriented vertically and the x-axis is oriented horizontally, the slope of the characteristics is $1/u(x,0)$, thus if the initial condition $u(x,0) = f(x)$ is a decreasing function of x in some interval, the slopes of the characteristics will increase, leading to the intersection of the characteristics at some $t > 0$. At the location and time of intersection of two characteristics with different values of u, a phenomenon known as a **shock** occurs.
- If the initial condition contains a jump discontinuity at some x_0 where $f(x_0^-) < f(x_0^+)$ there is a region between the characteristics $x(t) = f(x_0^-)t + x_0$ and $x(t) = f(x_0^+)t + x_0$ that appears to have no solution. This phenomenon is known as a **rarefaction**.

Quasilinear Cauchy IVP

$$u_t + c(u)u_x = 0 \text{ for } -\infty < x < \infty \text{ and } t > 0$$
$$u(x,0) = \begin{cases} u_L & \text{if } x < x_0 \\ u_R & \text{if } x > x_0 \end{cases}$$

Suppose $u_L > u_R$. Imagine the fluid to the left of x_0 is traveling faster than the fluid to the right of x_0 and thus is overtaking it resulting in a shock.

In order to illustrate the handling of a shock, rewrite the quasilinear partial differential in conservation form as

$$u_t + (C(u))_x = 0$$

where $C'(u) = c(u)$.

Rankine-Hugoniot Condition

Assume there is a smooth solution with the property that $u(x,t) = h(x - c_s t)$ where c_s denotes the speed of the shock and $\lim_{x\to-\infty} u(x,t) = u_L$ and $\lim_{x\to\infty} u(x,t) = u_R$.

$$-c_s h'(x - c_s t) + C(h(x - c_s t))' = 0$$

$$\int_{-\infty}^{\infty} [-c_s h'(x - c_s t) + C(h(x - c_s t))'] \, dx = 0$$

$$-c_s(u_R - u_L) + C(u_R) - C(u_L) = 0$$

$$c_s = \frac{C(u_R) - C(u_L)}{u_R - u_L}$$

The shock speed is a function of t. The last equation is often concisely written as

$$\frac{dx_s}{dt} = \frac{[C(u)]}{[u]}$$

and is known as the **Rankine-Hugoniot** condition.

Determining the Location of a Shock

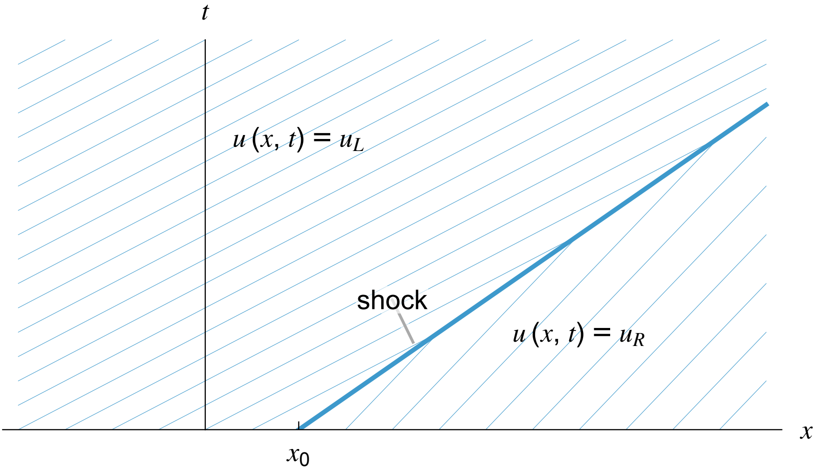
Consider the IVP:

$$\frac{dx_s}{dt} = \frac{[C(u)]}{[u]}$$
$$x_s(0) = x_0.$$

To the left of the shock, *i.e.*, where $x < x_s(t)$, the solution is defined as $u(x,t) = u_L$ while to the right of the shock $u(x,t) = u_R$. The shock itself is at

$$x_s(t) = \left(\frac{C(u_R) - C(u_L)}{u_R - u_L} \right) t + x_0.$$

Illustration



The typical solution to a quasilinear partial differential equation admitting a shock due to a jump discontinuity in the initial data.

Rarefaction

When $u_L < u_R$, the characteristics to the left of x_0 angle away from the characteristics to the right. As before the characteristics are described by the equations $x = c(f(x_i))t + x_i$ where f is a function denoting the initial data. This leaves a region between the lines $x = c(u_L)t + x_0$ and $x = c(u_R)t + x_0$ which appears to contain no characteristics. A solution does exist in this equation if the initial data is modified to include a set-valued condition. Suppose the initial condition is written as

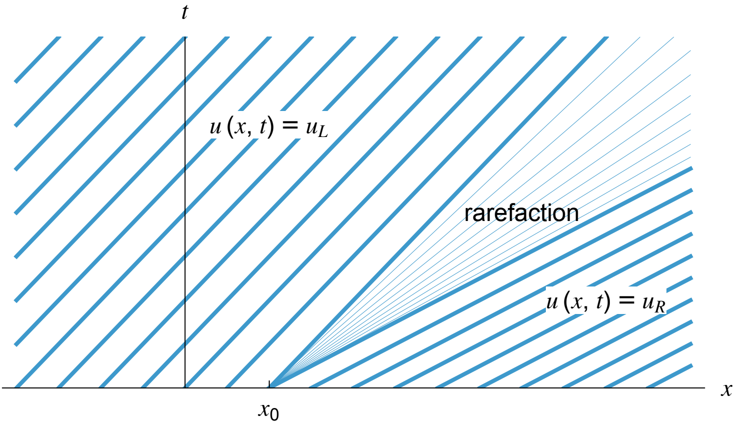
$$f(x) = \begin{cases} u_L & \text{if } x < x_0 \\ [u_L, u_R] & \text{if } x = x_0 \\ u_R & \text{if } x > x_0. \end{cases}$$

Intuitively the initial data takes on all values in the interval $[u_L, u_R]$ when $x = x_0$.

Illustration

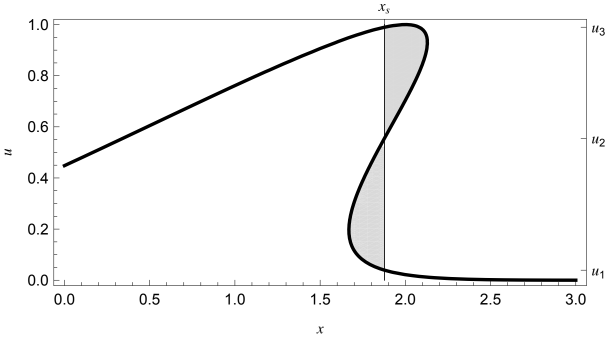
In the rarefaction region the characteristics all begin at x_0 and have the form

$$x = c(u)t + x_0 \text{ for } u_L < u < u_R.$$



Inviscid Burgers' Equation and Shocks

If the inviscid Burgers' equation is thought of as a model of fluid flow, then the downstream fluid is moving slower than the upstream fluid, or equivalently, the upstream fluid is overtaking the downstream fluid. Since the dependent variable u is constant along each characteristic, the intersection of characteristics implies u is multivalued, which means the solution is not well-defined.



Example

The shock will form at the earliest time the surface $u(x, t)$ has a vertical tangent. Using implicit differentiation on the general implicit solution to the inviscid Burgers' equation, the slope of the solution in the x-direction is

$$u_x = \frac{f'(z)}{1 + t\, f'(z)}.$$

A vertical tangent occurs when u_x is undefined, or equivalently when $t = -1/f'(z)$. Thus the earliest time this occurs is the time for which $f'(z)$ takes on its negative minimum.

Case: $f(x) = e^{-x^2}$

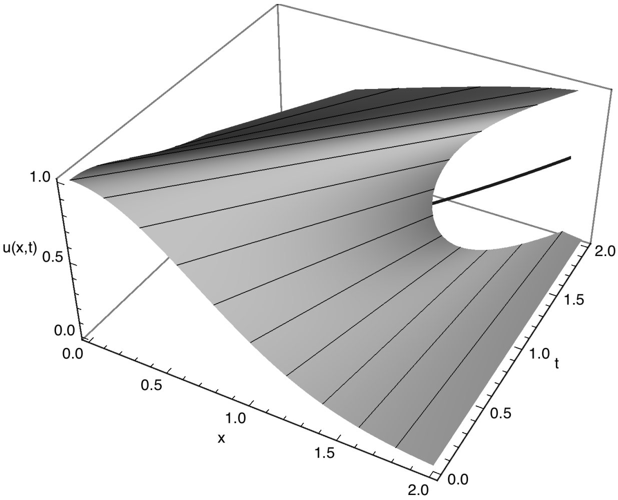
- The shock forms at $t_0 = \sqrt{e/2}$ when $f(x) = e^{-x^2}$.
- The x-coordinate at which the shock forms is found by solving the equation $1 + t_0 f'(x_0) = 0$ which makes u_x undefined and gives the coordinate on the $t = 0$ line where the characteristic passing through the beginning of the shock passes.
- Solving the equation,

$$0 = 1 - 2\sqrt{\frac{e}{2}}x_0 e^{-x_0^2} \implies x_0 = \frac{1}{\sqrt{2}}.$$

- The shock begins at the point with coordinates:

$$(x, t) = (x_0 + t_0 f(x_0), t_0) = \left(\sqrt{2}, \sqrt{\frac{e}{2}}\right)$$

Illustration



Locating the Shock

- Suppose the shock is located at position x_s at time t.
- The integral of u is conserved even when passing through the shock and since the shock has infinitesimal width, the definite integral must be zero.
- The correct value of u can be found by placing the shock at such as position that the shaded area to the right of x_s and left of the curve $u(x,t) = f(x - u(x,t)t)$ equals the shaded area to the left of x_s and right of the curve. Since one region is on either side of the vertical line and the regions are of equal area, their sum is 0.
- The value of u where x_s intersects the curve $u(x,t) = f(x - u(x,t)t)$ and which separates the left and right regions is then the "correct" value of $u(x,t)$.

Speed of the Shock

Using the conservation of the integral of u, by choosing $a < x_s$ and $b > x_s$ and setting $\nu = 0$, in the limit as $a \to x_s^-$ and $b \to x_s^+$,

$$u(x_s+, t)\frac{dx_s}{dt} - \frac{1}{2}(u(x_s+, t))^2 = u(x_s-, t)\frac{dx_s}{dt} - \frac{1}{2}(u(x_s-, t))^2$$

$$\frac{dx_s}{dt} = \frac{1}{2}(u(x_s+, t) + u(x_s-, t))$$

where $u(x_s+, t)$ and $u(x_s-, t)$ denote the values of the solution. Therefore the shock moves at a velocity which is the average of the values of u immediately on either side of the shock.