The Korteweg-de Vries Equation

Partial Differential Equations

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Objectives

In this lesson we will:

- present the Korteweg-de Vries (KdV) equation,
- explore the equivalence of several forms of the KdV equation, and
- find a soliton solution to the KdV equation.

Korteweg-de Vries Equation

The Korteweg-de Vries equation was initially studied as a mathematical model of the motion of waves in shallow water.

The Korteweg-de Vries equation has many equivalent mathematical formulations.

$$
\begin{aligned}
u_t + 6u\, u_x + u_{xxx} &= 0 \\
u_t + u\, u_x + k\, u_{xxx} &= 0 \\
u_t - 6u\, u_x + u_{xxx} &= 0 \\
u_t + u\, u_x + u_{xxx} &= 0 \\
u_t + \alpha u\, u_x + \beta u_{xxx} &= 0.
\end{aligned}
$$

Equivalence

Suppose $x = A\,X$, $t = B\,T$, and $u = C\,V$ then

$$u_t + 6u\,u_x + u_{xxx} = \frac{C}{B} V_T + \frac{6C^2}{A} V\,V_X + \frac{C}{A^3} V_{XXX} = 0,$$

which is equivalent to

$$V_T + \frac{6B\,C}{A} V\,V_X + \frac{B}{A^3} V_{XXX} = 0.$$

By choosing the constants A, B, and C appropriately, any form of the Korteweg-de Vries equation can be obtained.

Traveling Wave Solution

Set $\xi = x - c\,t$ and substitute $u(x,t) = U(\xi)$ into the Korteweg-de Vries equation.

$$\frac{d^3U}{d\xi^3} + 6U\frac{dU}{d\xi} - c\frac{dU}{d\xi} = 0.$$

Integrate both sides with respect to ξ.

$$\frac{d^2U}{d\xi^2} + 3U^2 - c\,U = A,$$

where A is an arbitrary constant of integration.

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Integrate both sides with respect to ξ.

$$\frac{d^2U}{d\xi^2} + 3U^2 - c\,U = A,$$

where A is an arbitrary constant of integration.

Assume that $U \to 0$, $U' \to 0$, and $U'' \to 0$ as $\xi \to \infty$ and as $\xi \to -\infty$.

$$\frac{d^2U}{d\xi^2} + 3U^2 - c\,U = 0.$$

Multiply the ordinary differential equation by $U' = dU/d\xi$ and integrate both sides of the resulting equation.

$$\frac{1}{2}\left(\frac{dU}{d\xi}\right)^2 + U^3 - \frac{1}{2}cU^2 = B,$$

where B is again an arbitrary constant of integration.

Traveling Wave Solution

The assumptions that $U \to 0$ and $U' \to 0$ as $\xi \to \infty$ and $\xi \to -\infty$ imply that $B = 0$, and therefore, after solving for $dU/d\xi$,

$$\frac{dU}{d\xi} = \pm U\sqrt{c - 2U},$$

where the $\pm$ sign should be chosen according to the sign of $dU/d\xi$.

In the following, the $\pm$ sign will be dropped since it makes no difference in the integration steps to follow.

Traveling Wave Solution

Separate the variables and integrate.

$$\int \frac{1}{U\sqrt{c-2U}}\, dU = \xi + A,$$

where A is again a constant.

To evaluate the integral on the left side, make a change of variable which, while not obvious, is motivated by identities from hyperbolic trigonometry,

$$U = \frac{c}{2}\operatorname{sech}^2\zeta \implies dU = -c\frac{\sinh\zeta}{\cosh^3\zeta}\, d\zeta.$$

The substitution results in

$$\begin{aligned}\int \frac{1}{U\sqrt{c-2U}}\, dU &= \int \frac{2}{c\operatorname{sech}^2\zeta\sqrt{c-c\operatorname{sech}^2\zeta}}\frac{-c\sinh\zeta}{\cosh^3\zeta}\, d\zeta \\ &= -\frac{2}{\sqrt{c}}\int 1\, d\zeta = -\frac{2}{\sqrt{c}}\zeta = -\frac{2}{\sqrt{c}}\operatorname{sech}^{-1}\sqrt{\frac{2U}{c}}.\end{aligned}$$

Traveling Wave Solution

Thus the equation,

$$-\frac{2}{\sqrt{c}}\,\mathrm{sech}^{-1}\sqrt{\frac{2U}{c}} = \xi + A$$

is solved for $U = U(\xi)$ to give

$$U(\xi) = \frac{c}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}\xi + A\right).$$

Therefore, the Korteweg-de Vries equation has the following traveling wave solution:

$$u(x,t) = \frac{c}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - c\,t) + A\right).$$

The presence of the term $\sqrt{c}$ and the desire to find a real-valued solution, suggest that $c > 0$.

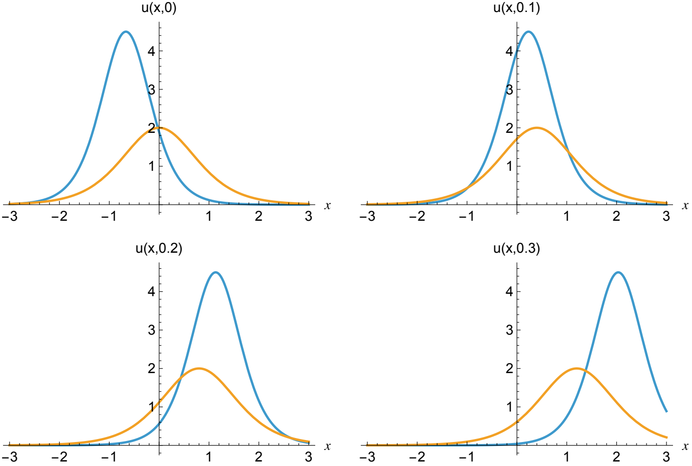
Traveling Wave Solution

If the constant c is replaced by $4c^2$ for convenience, then

$$u(x,t) = 2c^2 \operatorname{sech}^2(c(x - 4c^2 t) + A).$$

- This form of solution is a soliton solution of the Korteweg-de Vries equation.
- This soliton solution is positive for all x and t.
- For each fixed t it has a single maximum, and $u(x,t) \to 0$ as $|x| \to \infty$.
- All such solutions travel to the right with a speed depending on the amplitude of the solution.

Illustration



The blue soliton has $c = 3/2$ and $A = 1$ while the yellow soliton curve has $c = 1$ and $A = 0$. It is apparent the blue soliton is traveling faster to the right than the yellow soliton.

Other Solutions

If the assumption that the traveling wave solution and its derivatives vanish at infinity is relaxed, then the function $U(\xi)$ satisfies:

$$\frac{1}{2}\left(\frac{dU}{d\xi}\right)^2 = -U^3 + \frac{c}{2}U^2 + A\,U + B \equiv F(U)$$

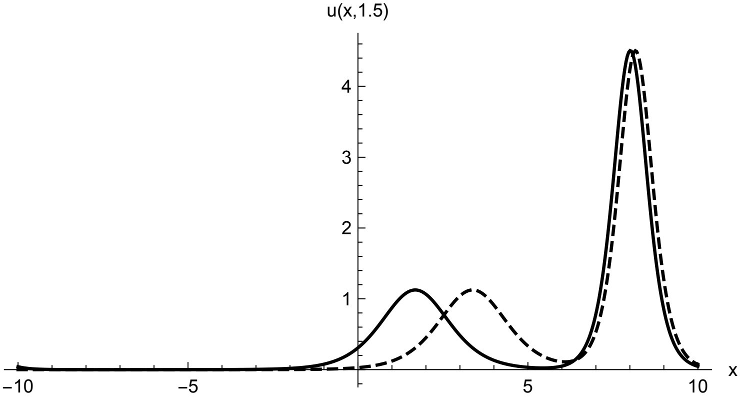
where A and B are arbitrary constants.

Solutions are real if and only if $F(U)$ is positive. Since the function $F(U)$ is a cubic polynomial (with a negative leading coefficient), it may have one or three real zeros.

Properties of Soliton Solutions

Soliton solutions maintain their form even after two solitons propagating at different speeds collide. The waves seem to simply pass through each other and each keeps its wave profile with only a slight phase shift in wave position (the peak of the wave profile).

Illustration



When solitons interact they experience a phase shift. The solid curve depicts the profile of the solution to an initial value problem in which interaction takes place while the dashed curve shows the sum of the solitons had there been no interaction.