Nonlinear Schrödinger Equation

Partial Differential Equations

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Objectives

In this lesson we will:

- introduce the Schrödinger wave equation, and
- find soliton solutions to the Schrödinger wave equation.

Standard Form of Nonlinear Schrödinger Equation

$$i\,\psi_t + \psi_{xx} + \gamma|\psi|^2\psi = 0,$$

where $i = \sqrt{-1}$ is the imaginary unit and ψ is a complex-valued function of t and x.

Assume a plane wave solution of the form

$$\psi(x,t) = A\,e^{i(kx-\omega t)},$$

where A, k, and ω are constants to be determined.

Plane Wave Solution

Recall: the constant k is called the wave number and ω is the frequency.

Substitute the plane wave function in the nonlinear Schrödinger equation to produce

$$i(-i\,\omega) + i^2 k^2 + \gamma\,A^2 = 0,$$

which can be simplified as

$$\omega = k^2 - \gamma\,A^2.$$

Therefore, for any k the function

$$\psi(x,t) = Ae^{i(kx-(k^2-\gamma A^2)t)}$$

is a solution to the nonlinear Schrödinger equation.

General Solution

A more general solution to the nonlinear Schrödinger equation takes the form

$$\psi(x,t) = f(x - c\,t)e^{i(kx-\omega t)}$$

and the symbols k, c, and ω are constants. Note that this solution incorporates features of a traveling wave and a plane wave. Substitute this solution into the nonlinear Schrödinger equation,

$$\begin{aligned} 0 &= -c\,if' + \omega f + f'' + 2i\,kf' - k^2 f + \gamma f^3 \\ &= f'' + i(2k - c)f' + (\omega - k^2)f + \gamma f^3. \end{aligned}$$

Let $\xi = x - c\,t$ and assume f is real-valued. This implies $c = 2k$ and

$$f'' + (\omega - k^2)f + \gamma f^3 = 0.$$

Solving for Function $f(\xi)$

$$f'' + (\omega - k^2)f + \gamma f^3 = 0.$$

Multiply by $2f'$ and antidifferente.

$$(f')^2 = A - (\omega - k^2)(f)^2 - \frac{\gamma}{2}(f)^4,$$

where A is an arbitrary constant of integration.

If the solution $\psi(x,t)$ and its derivative $\psi_x(x,t)$ vanish as $x \to \pm\infty$, then $f \to 0$ and $f' \to 0$ as $x \to \pm\infty$. This implies $A = 0$ and function f satisfies

$$(f')^2 = (k^2 - \omega)(f)^2 - \frac{\gamma}{2}(f)^4.$$

Solving for f' produces

$$\frac{df}{d\xi} = \pm|f|\sqrt{(k^2 - \omega) - \frac{\gamma}{2}(f)^2}.$$

Separate Variables

$$\frac{df}{d\xi} = \pm|f|\sqrt{(k^2-\omega) - \frac{\gamma}{2}(f)^2}.$$

Separate variables in this ODE and integrate both sides imply (taking the negative root),

$$\xi + A = \int \frac{-1}{|f|\sqrt{(k^2-\omega) - \frac{\gamma}{2}(f)^2}}\, df$$

where A is an arbitrary constant of integration.

Make the change of variable,

$$f = \sqrt{\frac{2(k^2-\omega)}{\gamma}}\,\mathrm{sech}\,\zeta \implies df = -\sqrt{\frac{2(k^2-\omega)}{\gamma}}\,\mathrm{sech}\,\zeta\,\tanh\zeta\, d\zeta.$$

The indefinite integral is rewritten as

$$\int \frac{-1}{|f|\sqrt{(k^2-\omega) - \frac{\gamma}{2}(f)^2}}\, df = \int \frac{1}{\sqrt{k^2-\omega}}\, d\zeta = \frac{1}{\sqrt{k^2-\omega}}\zeta$$

$$= \frac{1}{\sqrt{k^2-\omega}}\,\mathrm{sech}^{-1}\left(\sqrt{\frac{\gamma}{2(k^2-\omega)}}f\right).$$

Solving for Function $\psi(x, t)$

$$\frac{1}{\sqrt{k^2 - \omega}} \operatorname{sech}^{-1}\left(\sqrt{\frac{\gamma}{2(k^2 - \omega)}}\, f\right) = \xi + A$$

Solve for $f(\xi)$.

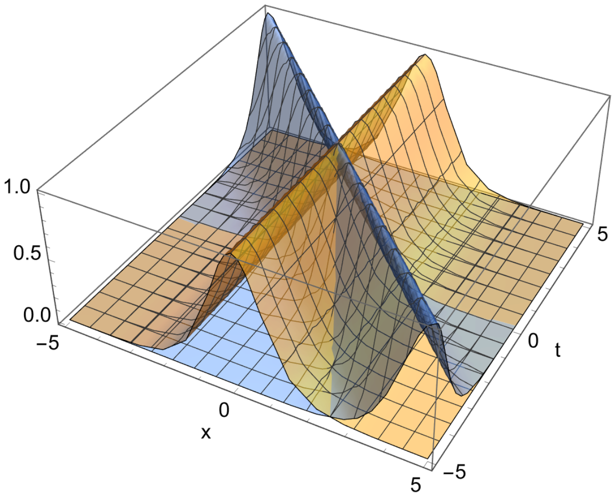
$$f(\xi) = \sqrt{\frac{2(k^2 - \omega)}{\gamma}} \operatorname{sech}\left((k^2 - \omega)^{1/2}(\xi + A)\right)$$

Therefore, the following is a solution of the nonlinear Schrödinger equation,

$$\psi(x, t) = \sqrt{\frac{2(k^2 - \omega)}{\gamma}} e^{i(kx - \omega t)} \operatorname{sech}\left((k^2 - \omega)^{1/2}(x - 2k\,t + A)\right).$$

The solution found above is called an **envelope soliton solution**.

Illustration



The squared amplitude of two soliton solutions ($\psi_1(x,t) = e^{it} \operatorname{sech} x$ and $\psi_2(x,t) = e^{i(-x/2+3t/4)} \operatorname{sech}(x+t)$) to the nonlinear Schrödinger equation pass each other without constructive or destructive interference.