Traveling Waves

Partial Differential Equations

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Objectives

In this lesson we will:

- introduce the traveling wave form of a solution to a PDE, and
- use the traveling wave to solve some PDEs.

Traveling Wave

If $u(x, t)$ represents a disturbance or signal strength at location x at time t, it is a **traveling wave** if u has the form,

$$u(x, t) = U(x - c\,t)$$

for some single-variable function U and a fixed real constant c.

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The solution of the standard wave equation on the real number line with the initial conditions

$$u(x,0) = f(x) \text{ and } u_t(x,0) = 0,$$

(the plucked string) is given by

$$u(x,t) = \frac{1}{2}(f(x - c\,t) + f(x + c\,t)).$$

- $u(x,t)$ is the sum of two simple traveling waves, the initial disturbance $f(x)$ of the string is propagated to the right and left with speed $|c|$, each at half of the magnitude.
- Strictly speaking, $u(x,t)$ is not a traveling wave itself, it is the sum of two simple traveling waves.

Example

Find all bounded, traveling wave solutions of the (linear) partial differential equation,

$$u_t - u_{xxx} = 0.$$

Solution (1 of 2)

Set $\xi = x - c\,t$ and substitute $u(x,t) = U(\xi)$ in the equation,

$$-c\,U'(\xi) - U'''(\xi) = 0.$$

Integrating the above equation produces

$$U''(\xi) + c\,U(\xi) = A,$$

where A is an arbitrary constant of integration. If $c < 0$, the general solution is

$$U(\xi) = A_1 e^{\sqrt{-c}\,\xi} + A_2 e^{-\sqrt{-c}\,\xi} + \frac{A}{c}$$

Where A_1 and A_2 are arbitrary constants.

Remark: unless A_1 and A_2 are both zero, $U(\xi)$ is unbounded.

Solution (2 of 2)

If $c = 0$, there is no bounded solution other than constant solutions.

If $c > 0$, the general solution is

$$U(\xi) = A_1 \cos(\sqrt{c}\,\xi) + A_2 \sin(\sqrt{c}\,\xi) + \frac{A}{c}$$

where A_1 and A_2 are again arbitrary constants.

The function $U(\xi)$ is bounded and leads to the following bounded traveling wave solution,

$$u(x - c\,t) = A_1 \cos(\sqrt{c}(x - c\,t)) + A_2 \sin(\sqrt{c}(x - c\,t)) + \frac{A}{c}$$

for any choices of constants A_1, A_2, and A.

Remarks

- The PDE of the previous example is a linear, dissipative partial differential equation.
- The nonconstant, bounded traveling wave solutions occur only for $c > 0$. This implies that all traveling wave solutions travel to the right only.
- Disregarding the constant A/c, all nonconstant traveling wave solutions are linear combinations of

$$u_1(x,t) = \cos(\sqrt{c}(x - c\,t)) \text{ and } u_2(x,t) = \sin(\sqrt{c}(x - c\,t)).$$

- Setting $k = \sqrt{c}$ and $\omega = k^3$, these two solutions can be written as

$$u_1(x,t) = \cos(kx - \omega t) \text{ and } u_2(x,t) = \sin(kx - \omega t).$$

Remarks

- Such traveling wave solutions are often called one-dimensional **plane wave solutions** and are often conveniently represented in the form of complex exponentials as $u(x,t) = e^{i(kx-\omega t)}$ and its complex conjugate.
- The constant k is called the **wave number** and ω is the **the angular frequency**.
- The equality $\omega = k^3$ describes the relationship between the angular frequency and the wave number and is often referred to as the **dispersion relation**.

Free Schrödinger Equation

Consider the partial differential equation

$$\psi_t = i\,\psi_{xx}$$

referred to as the free Schrödinger equation for free particles.

If $\psi(x,t) = e^{i(kx-\omega t)}$ is a solution, the following dispersion relation holds,

$$-i\,\omega = -i\,k^2 \iff \omega = k^2.$$

Solitons

- A class of traveling wave solution, of particular importance for nonlinear partial differential equations, is the **solitary wave** or **soliton**, solution which is a special type of traveling wave solution.
- A solitary wave solution has two important features.
  - It must have a permanent form, meaning that its profile will not change as it propagates with time.
  - It is localized, meaning that it decays to 0 or approaches a constant at infinity.
- A soliton solution is a solitary wave solution that is stable in the sense that it maintains its wave profile even after colliding with another soliton.