Discretization of Initial and Boundary Conditions

Partial Differential Equations

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Fall 2025

Objectives

In this lesson we will:

- develop discrete approximations to Dirichlet, Neumann, and Robin boundary conditions, and
- develop discrete approximations to initial conditions for initial boundary value problems.

Initial Conditions

Suppose the initial condition is described as

$$u(x, 0) = f(x) \text{ for } 0 < x < L.$$

The interval $[0, L]$ is partitioned into N equally sized subintervals of width $h = L/N$ with end points $x_i = i\,h$ for $i = 0, 1, \ldots, N$.

Discretizing the initial condition is straight forward as $u_i^0 = u(i\,h, 0) = f(i\,h)$ for $i = 0, 1, \ldots, N$.

Dirichlet Boundary Conditions

Suppose the initial conditions are described by the equations:

$$
\begin{aligned}
u(0, t) &= a(t) \\
u(L, t) &= b(t)
\end{aligned}
$$

for $t > 0$. The interval $[0, L]$ is partitioned into N equally sized subintervals of width $h = L/N$ with end points $x_i = i\,h$ for $i = 0, 1, \ldots, N$.

The Dirichlet boundary conditions are handled by assigning the known values at the endpoints of the interval, $u_0^j = u((0)h, j\,k) = a(j\,k)$ and $u_N^j = u(N\,h, j\,k) = b(j\,k)$ for $j \in \mathbb{N}$.

Robin Boundary Conditions

Suppose the boundary conditions are:

$$K_0 u_x(0,t) = \alpha(u(0,t) - a(t))$$
$$K_0 u_x(L,t) = -\beta(u(L,t) - b(t)),$$

for $t > 0$ where K_0, α, and β are positive constants and $a(t)$ and $b(t)$ are known functions.

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At $x = 0$ a forward difference formula for the partial derivative with respect to x is employed. For $j \in \mathbb{N}$,

$$K_0 \frac{(u_1^j - u_0^j)}{h} = \alpha(u_0^j - a(j\,k)).$$

At $x = L$ a backward difference formula is used, which when written as

$$K_0 \frac{(u_{N-1}^j - u_N^j)}{h} = \beta(u_N^j - b(j\,k)),$$

eliminates the minus sign from the right-hand side of the boundary condition. These are $O(h)$ approximations to the boundary conditions.

Fictitious Points

If two fictitious points at $x = -h$ and $x = L + h$ are introduced, the $O(h^2)$ accurate centered difference formula can be used. In this case the discretized Robin boundary conditions can be expressed as

$$
\begin{aligned}
K_0 \frac{(u_1^j - u_{-1}^j)}{2h} &= \alpha(u_0^j - a(j\,k)) \\
K_0 \frac{(u_{N+1}^j - u_{N-1}^j)}{2h} &= -\beta(u_N^j - b(j\,k))
\end{aligned}
$$

Neumann Boundary Conditions

Recall that as $\alpha \to 0$ and $\beta \to 0$ the Robin boundary conditions asymptotically approach homogeneous Neumann boundary conditions.