Derivations of Laplace's Equation

Partial Differential Equations

J Robert Buchanan

Department of Mathematics

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Objectives

In this lesson we will examine a time-independent partial differential equation known as Laplace's equation. It often arises as

- a steady-state solution to the heat or wave equation, and
- in as a potential for a vector field.

Steady-State Solutions

Consider the initial, boundary value problems for the heat and wave equations.

Heat Equation

$$\begin{aligned} u_t &= \kappa u_{xx} + Q(x,t) \text{ for } 0 < x < L,\ t > 0 \\ K_0 u_x(0,t) &= \alpha(u(0,t) - g_1(t)) \text{ for } t > 0 \\ K_0 u_x(L,t) &= -\beta(u(L,t) - g_2(t)) \text{ for } t > 0 \\ u(x,0) &= f(x) \text{ for } 0 < x < L \end{aligned}$$

Wave Equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + F(x,t) \text{ for } 0 < x < L,\ t > 0 \\ T_0 u_x(0,t) &= k_1(u(0,t) - d_1(t)) \text{ for } t > 0 \\ T_0 u_x(L,t) &= -k_2(u(L,t) - d_2(t)) \text{ for } t > 0 \\ u(x,0) &= f(x) \text{ for } 0 < x < L \\ u_t(x,0) &= g(x) \text{ for } 0 < x < L \end{aligned}$$

A t-independent solution to the PDE and boundary conditions is known as a steady-state solution.

Steady-State Solutions

For either the heat or wave equation if $u(x,t) = U(x)$, then we have:

Heat Equation	Wave Equation
$0 = \kappa U_{xx} + Q(x,t)$ for $0 < x < L$	$0 = c^2 U_{xx} + F(x)$ for $0 < x < L$
$K_0 U_x(0) = \alpha U(0)$	$T_0 U_x(0) = k_1 U(0)$
$K_0 U_x(L) = -\beta U(L)$	$T_0 U_x(L) = -k_2 U(L)$

In either case the differential equation can be written in the general form as

$$U''(x) + f(x) = 0 \text{ for } 0 < x < L.$$

- If $f(x) \equiv 0$ this is the one-dimensional Laplace's equation.
- If $f(x) \not\equiv 0$ this is the one-dimensional Poisson's equation.

Higher Dimensional Versions

In two dimensions,

$$\Delta u = u_{xx} + u_{yy} = f(x, y) \text{ for } (x, y) \in \Omega \subset \mathbb{R}^2.$$

In three dimensions,

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = f(x, y, z) \text{ for } (x, y, z) \in \Omega \subset \mathbb{R}^3.$$

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$$\Delta u = u_{xx} + u_{yy} + u_{zz} = f(x, y, z) \text{ for } (x, y, z) \in \Omega \subset \mathbb{R}^3.$$

If the right-hand side of the equation is 0 on Ω this is Laplace's equation, otherwise it is Poisson's equation.

Boundary Conditions

Suppose

$$\Delta u = u_{xx} + u_{yy} = f(x,y) \text{ for } (x,y) \in \Omega \subset \mathbb{R}^2,$$

where Ω is an open, connected set.

Dirichlet boundary conditions can be expressed as

$$u(x,y) = \phi(x,y) \text{ for } (x,y) \in \partial\Omega$$

where $\partial\Omega$ denotes the boundary of curve of Ω.

Neumann boundary conditions can be expressed as

$$\nabla u(x,y) \cdot \mathbf{n} = \psi(x,y) \text{ for } (x,y) \in \partial\Omega$$

where **n** is the unit outward normal vector to $\partial\Omega$.

Boundary Conditions

Suppose

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = f(x, y, z) \text{ for } (x, y, z) \in \Omega \subset \mathbb{R}^3,$$

where Ω is an open, connected set.

Dirichlet boundary conditions can be expressed as

$$u(x, y, z) = \phi(x, y, z) \text{ for } (x, y, z) \in \partial\Omega$$

where $\partial\Omega$ denotes the boundary of surface of Ω.

Neumann boundary conditions can be expressed as

$$\nabla u(x, y, z) \cdot \mathbf{n} = \psi(x, y, z) \text{ for } (x, y, z) \in \partial\Omega$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$.