Applications of First-Order PDEs

Partial Differential Equations

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Objectives

In this lesson we will apply the solution techniques learned earlier to finding the solutions to some first-order PDEs arising in the natural and physical sciences.

Structured Population Growth

- Consider a population consisting of individuals of different ages or sizes x at time t.
- Define $p(x,t)$ to be age-density of the population. The number of individuals in the population between ages x and $x + \Delta x$ at time t is approximately
$$p(x,t)\,\Delta x.$$
- Define $N(t)$ to be the total population at time t,
$$N(t) = \int_0^\infty p(x,t)\,dx.$$

We wish to develop a PDE model for the evolution of the age-density.

Calendar Time and Age

- Population with ages in $[x, x + \Delta x]$ at time $t + \Delta t$ is approximately $p(x, t + \Delta t)\,\Delta x$.
- If individuals age at the same rate as time passes then

$$p(x, t + \Delta t)\,\Delta x = p(x - \Delta t, t)\,\Delta x$$

  in other words, if deaths are ignored the number of individuals with ages $[x, x + \Delta x]$ at time $t + \Delta t$ is the same as the number of individuals with ages $[x - \Delta t, x]$ at time t.
- Let $\mu(x)$ the rate of death (deaths per unit time) of individuals of age x. The number of deaths in the population of individuals with ages $[x - \Delta t, x]$ between times t and $t + \Delta t$ is approximately

$$\mu(x - \Delta t)p(x - \Delta t, t)\,\Delta x\,\Delta t.$$

Population Equation

$$p(x, t + \Delta t)\,\Delta x \approx p(x - \Delta t, t)\,\Delta x - \mu(x - \Delta t)p(x - \Delta t, t)\,\Delta x\,\Delta t$$

$$p(x, t + \Delta t) - p(x - \Delta t, t) \approx -\mu(x - \Delta t)p(x - \Delta t, t)\,\Delta t$$

Add and subtract $p(x, t)$ on the left-hand side of the approximation.

$$p(x, t + \Delta t) - p(x, t) + p(x, t) - p(x - \Delta t, t) \approx -\mu(x - \Delta t)p(x - \Delta t, t)\,\Delta t$$

$$\frac{p(x, t + \Delta t) - p(x, t)}{\Delta t} + \frac{p(x, t) - p(x - \Delta t, t)}{\Delta t} \approx -\mu(x - \Delta t)p(x - \Delta t, t)$$

$$p_t + p_x = -\mu(x)\,p$$

von Foerster Equation

We have the first-order linear PDE

$$p_t + p_x = -\mu(x)\, p$$

with side condition

$$p(x, 0) = f(x),$$

specifying the initial distribution of the ages within the population, and

$$p(0, t) = \phi(t)$$

the density of new births at time t.

Solution (1 of 2)

- Using the method of characteristics,

$$\frac{dx}{dt} = 1 \implies x - t = k$$

where k is a constant.

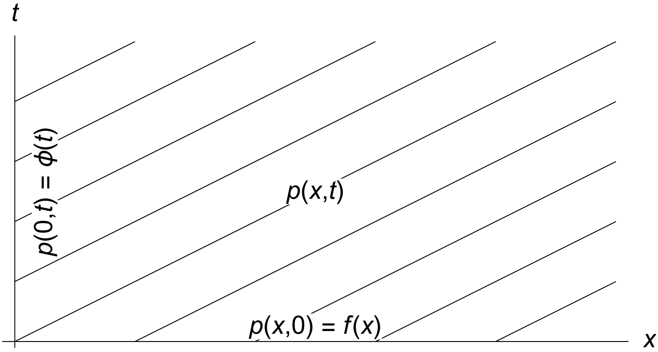
- Assuming $p(x,t) = p(x(t),t) = p(t)$ then

$$\frac{dp}{dt} = p_t + p_x \frac{dx}{dt} = p_t + p_x = -\mu(x)\, p = -\mu(t+k)p.$$

- Solving this first-order linear ODE yields,

$$p(t) = C(k) e^{-\int_{t_0}^{t} \mu(s+k)\, ds}.$$

Characteristics



$$p(t) = C(k)e^{-\int_{t_0}^{t} \mu(s+k)\,ds}.$$

Function $p(t)$ is defined for $t \geq 0$ when $k \geq 0$ and is defined for $t \geq -k$ when $k < 0$.

Solution (2 of 2)

Apply side conditions.

$$p(0) = C(k) = \begin{cases} f(k) & \text{if } k \geq 0, \\ \phi(-k) & \text{if } k < 0. \end{cases}$$

Thus the solution can be expressed as

$$p(x,t) = \begin{cases} f(x-t)e^{\int_0^t \mu(s+x-t)\,ds} & \text{if } k \geq 0, \\ \phi(t-x)e^{-\int_{-k}^t \mu(s+x-t)\,ds} & \text{if } k < 0. \end{cases}$$

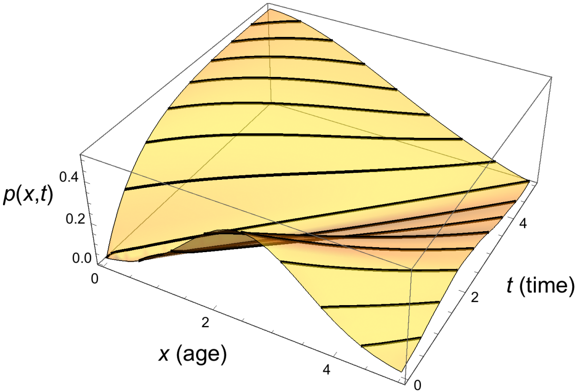
$$= \begin{cases} f(x-t)e^{-\int_{x-t}^x \mu(s)\,ds} & \text{if } t < x, \\ \phi(t-x)e^{-\int_0^x \mu(s)\,dx} & \text{if } t > x. \end{cases}$$

Example

$$\phi(t) = \frac{e^{-25/8}}{\sqrt{2\pi}} + \frac{1}{2}(1 - e^{-t})$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-5/2)^2/2}$$

$$\mu(x) = 1 - e^{-x/5}$$



Comments

A reasonable assumption is that new births at time t depend on the distribution of ages in the population and/or the total population.

If the birth rate is age-dependent denote it as $\beta(x)$ and then,

$$p(0,t) = \int_0^\infty \beta(x) p(x,t)\, dx.$$

This is called as **nonlocal boundary condition** since $p(0,t)$ depends on the unknown density $p(x,t)$.

Nonlocal Boundary Condition

$$\begin{aligned}
\phi(t) = p(0,t) &= \int_0^\infty \beta(x)p(x,t)\,dx \\
&= \int_0^t \beta(x)p(x,t)\,dx + \int_t^\infty \beta(x)p(x,t)\,dx \\
&= \int_0^t \beta(x)\phi(t-x)e^{-\int_0^x \mu(s)\,ds}\,dx + \int_t^\infty \beta(x)f(x-t)e^{-\int_{x-t}^x \mu(s)\,ds}\,dx
\end{aligned}$$

Volterra Integral Equation

Define $\psi(t) = \int_t^\infty \beta(x) f(x-t) e^{-\int_{x-t}^x \mu(s)\,ds}\,dx$, then

$$\phi(t) = \int_0^t \beta(x)\phi(t-x)e^{-\int_0^x \mu(s)\,ds}\,dx + \psi(t)$$

which is an example of a **Volterra integral equation**.

If the integral equation can be solved for $\phi(t)$ then, as before,

$$p(x,t) = \begin{cases} f(x-t)e^{-\int_{x-t}^x \mu(s)\,ds} & \text{if } t < x, \\ \phi(t-x)e^{-\int_0^x \mu(s)\,dx} & \text{if } t > x. \end{cases}$$

Traffic Flow

Consider a one-way, single lane road and the vehicles on it.

- Vehicles are not allowed to pass each other.
- The road has no exits or entrances. All vehicles on the road are already present. No additional vehicles will be added and none may leave.
- Function $\rho(x, t)$ denotes the density of vehicles, the number of vehicles per unit length of road (for example, cars/mile) at time t at the location x.
- Function $q(x, t)$ denotes the flow of traffic, the number of vehicles passing location x and at time t.

Model Development (1 of 3)

In consider the interval $[a, b]$ along the road. The total number of vehicles in interval $[a, b]$ given by

$$N(t) = \int_a^b \rho(x, t)\, dx.$$

The time rate of change of $N(t)$ is

$$\frac{dN}{dt} = \frac{d}{dt}\left[\int_a^b \rho(x, t)\, dx\right] = q(a, t) - q(b, t).$$

Question: what is an intuitive interpretation of the last equation?

Model Development (2 of 3)

$$q(a,t) - q(b,t) = -\int_a^b \frac{\partial}{\partial x}\left[q(x,t)\right]\,dx$$
$$= \frac{d}{dt}\left[\int_a^b \rho(x,t)\,dx\right]$$

Equating the right-hand sides and re-arranging terms produces:

$$\int_a^b \left(\frac{\partial}{\partial t}\left[\rho(x,t)\right] + \frac{\partial}{\partial x}\left[q(x,t)\right]\right)\,dx = 0,$$

which implies (since $[a,b]$ is arbitrary),

$$\rho_t + q_x = 0.$$

Model Development (3 of 3)

The PDE:

$$\rho_t + q_x = 0$$

contains 2 unknown functions. If we assume $q(x,t) = \rho(x,t)u(\rho(x,t))$ where u is the velocity function for a vehicle, then we have

$$\rho_t + (\rho\, u)_x = 0$$

which contains only the unknown traffic density.

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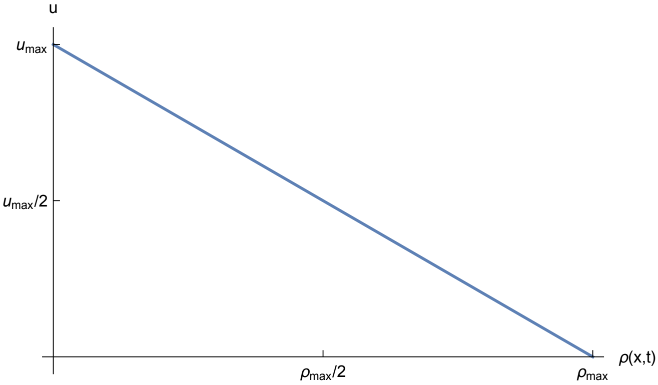
A common choice for u is

$$u = u_{\max}\left(1 - \frac{\rho(x,t)}{\rho_{\max}}\right).$$

Vehicle Speed *vs.* Density

$$u = u_{max}\left(1 - \frac{\rho(x,t)}{\rho_{max}}\right).$$



Quasilinear PDE

Using the common choice for u then

$$\begin{aligned} \rho_t + (\rho\, u)_x &= 0 \\ \rho_t + u_{\max}\left[\rho\left(1 - \frac{\rho}{\rho_{\max}}\right)\right]_x &= 0 \\ \rho_t + u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)\rho_x &= 0, \end{aligned}$$

which is a first-order quasilinear PDE.

We can solve this PDE with side condition $\rho(x, 0) = f(x)$.

Characteristic System

$$\frac{dt}{d\tau} = 1$$
$$\frac{dx}{d\tau} = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)$$
$$\frac{d\rho}{d\tau} = 0$$

Characteristic System

$$\frac{dt}{d\tau} = 1 \implies t(\tau) = \tau + A$$

$$\frac{dx}{d\tau} = u_{\text{max}} \left(1 - \frac{2\rho}{\rho_{\text{max}}}\right)$$

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$$\frac{dt}{d\tau} = 1 \implies t(\tau) = \tau + A$$

$$\frac{dx}{d\tau} = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right) \implies x(\tau) = u_{\max}\left(1 - \frac{2B}{\rho_{\max}}\right)\tau + C$$

$$\frac{d\rho}{d\tau} = 0 \implies \rho(\tau) = B$$

where A, B, and C are constants.

Characteristic Curves

- The initial density defined a non-characteristic curve $(0, s, f(s))$.
- Suppose the characteristics intersect this curve at $\tau = 0$.

$$
\begin{aligned}
t(0) &= 0 = A \\
x(0) &= s = C \\
\rho(0) &= f(s) = B
\end{aligned}
$$

which implies

$$
\begin{aligned}
t &= \tau \\
x &= u_{\max}\left(1 - \frac{2f(s)}{\rho_{\max}}\right)\tau + s \\
\rho &= f(s).
\end{aligned}
$$

Elimination of τ

Since $t = \tau$,

$$x = u_{\max}\left(1 - \frac{2f(s)}{\rho_{\max}}\right)t + s$$
$$\rho = f(s).$$

Consider the characteristics in the xt-plane.

- The characteristics are straight lines.
- The x-intercept lies at $(s, 0)$.
- The density of vehicles is constant (with value $f(s)$) along this characteristic.
- If $f(s) = f_0$ a constant, then all characteristics are parallel.

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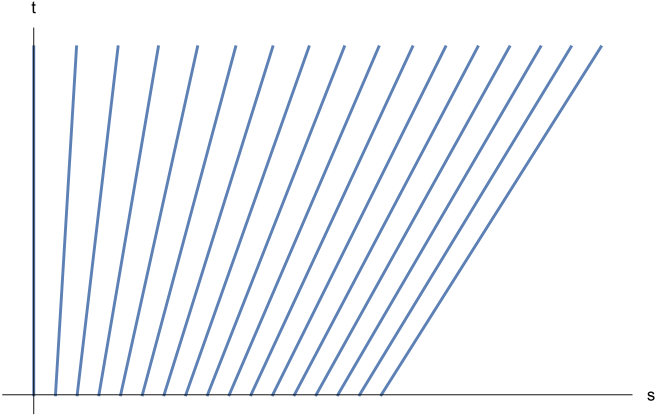
Question: what if $f(s)$ is not constant?

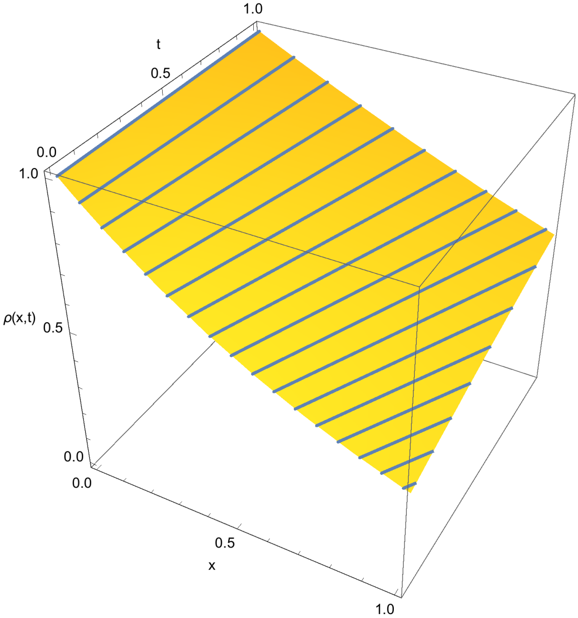
Case: $f'(s) < 0$

If $f(s)$ is decreasing then characteristics do not intersect for $t \geq 0$.



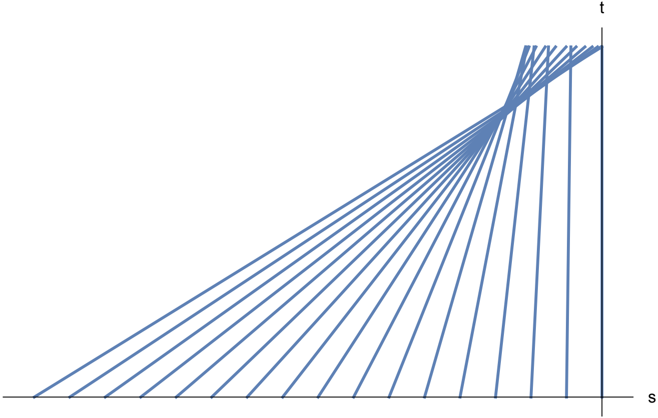
The density of vehicles is well-defined for $t \geq 0$.

Case: $f'(s) < 0$



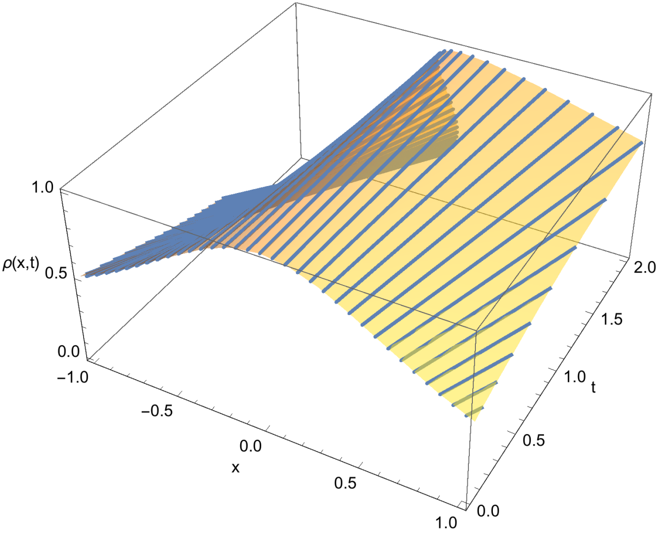
Case: $f'(s) > 0$

If $f(s)$ is increasing then characteristics intersect for some $t \geq 0$.

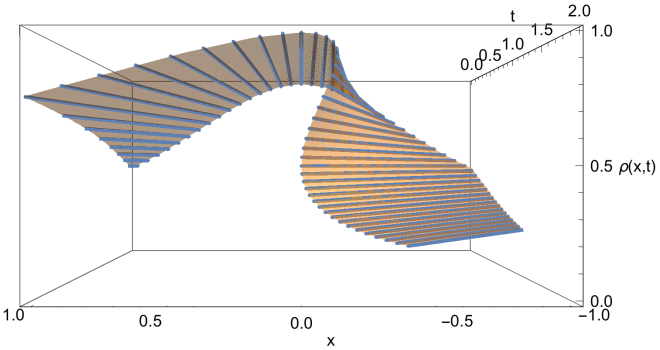


This implies the density takes on two different values. This phenomenon is known as a **shock**.

Case: $f'(s) > 0$



Case: $f'(s) > 0$



Question: when does the shock occur?

Time of Shock (1 of 2)

- The shock occurs at the earliest time for which there is a vertical tangent line to the integral surface.

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- Suppose the shock occurs at $t = t_0$, then

$$
\begin{aligned}
x(s) &= u_{\max}\left(1 - \frac{2f(s)}{\rho_{\max}}\right) t_0 + s \\
\rho(s) &= f(s).
\end{aligned}
$$

We must find the tangent vector to this parametrically defined curve.

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  We must find the tangent vector to this parametrically defined curve.
- Using the chain rule for derivatives,

$$\frac{d\rho}{dx} = \frac{d\rho/ds}{dx/ds} = \frac{f'(s)}{1 - \frac{2u_{\max} t_0}{\rho_{\max}} f'(s)}.$$

Time of Shock (2 of 2)

$$\frac{d\rho}{dx} = \frac{f'(s)}{1 - \frac{2u_{max}t_0}{\rho_{max}} f'(s)}$$

Thus the shock occurs when $f'(s) \neq 0$ and

$$1 - \frac{2u_{max}t_0}{\rho_{max}} f'(s) = 0 \iff t_0 = \frac{\rho_{max}}{2u_{max}f'(s)}.$$

Example

Suppose $\rho(x, 0) = f(x) = \dfrac{\rho_{\text{max}}/3}{1 + x^2}$.

1. Where is the point of maximum density initially?
2. Where is the point of maximum density at time $t > 0$?
3. When does the first shock occur?
4. Where does the first shock occur?

Solution (1 of 2)

$$f(s) = \frac{\rho_{\text{max}}/3}{1+s^2}$$

$$f'(s) = -\frac{2\rho_{\text{max}}}{3} \frac{x}{(1+s^2)^2}$$

At $t = 0$ the maximum density is at $x_0 = 0$.

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At $t = 0$ the maximum density is at $x_0 = 0$.

Recall: for each constant x_0 the solution has constant value $f(x_0)$ along the line

$$x = u_{\max} \left(1 - \frac{2f(x_0)}{\rho_{\max}}\right) t + x_0.$$

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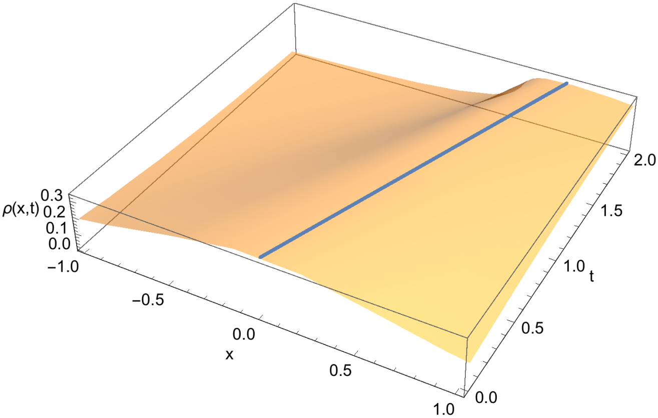
$$x = u_{\max}\left(1 - \frac{2f(x_0)}{\rho_{\max}}\right) t + x_0.$$

Thus if $x_0 = 0$ the maximum density at $t > 0$ is at

$$x = u_{\max}\left(1 - \frac{2f(0)}{\rho_{\max}}\right) t = \frac{u_{\max} t}{3}.$$

The point of maximum density is moving to the right at $1/3$ the maximum speed of the vehicles.

Tracking the Maximum Density



Solution (2 of 2)

The first shock arises when

$$t_0 = \frac{\rho_{\max}}{2u_{\max} f'(s)}.$$

The maximum of $f'(s)$ occurs at $s = -1/\sqrt{3}$ and thus

$$t_0 = \frac{\rho_{\max}}{2u_{\max} f'(-1/\sqrt{3})} = \frac{4}{u_{\max}\sqrt{3}},$$

is the time of the first shock.

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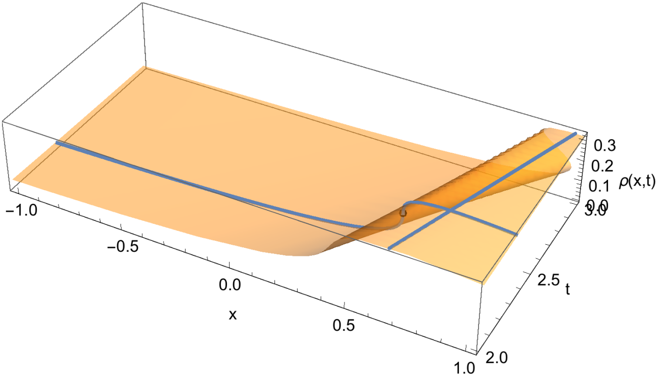
$$t_0 = \frac{\rho_{\max}}{2u_{\max} f'(-1/\sqrt{3})} = \frac{4}{u_{\max}\sqrt{3}},$$

is the time of the first shock.

The location of the first shock is

$$\begin{aligned} x &= u_{\max}\left(1 - \frac{2f(s)}{\rho_{\max}}\right) t + s \\ &= u_{\max}\left(1 - \frac{2f(-1/\sqrt{3})}{\rho_{\max}}\right) \frac{4}{u_{\max}\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}. \end{aligned}$$

Illustration



Note: the shock occurs at a level below the maximum density.