First-Order Linear PDEs

MATH 467 *Partial Differential Equations*

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Objectives

In this lesson we will learn:

- to classify first-order partial differential equations as either linear or quasilinear,
- to solve linear first-order partial differential equations,

Transport Equations

The general form of a first-order scalar PDE is

$$u_x + \nabla \cdot f(x, y, u, \nabla u) = g(x, y, u)$$

where

- $\nabla \cdot f$ denotes the divergence of f, and
- ∇u denotes the gradient of u.

Classification

A first-order PDE is **linear** if it can be written as

$$a(x, y)u_x(x, y) + b(x, y)u_y(x, y) = c(x, y)u(x, y) + d(x, y).$$

A first-order PDE is **semilinear** if it can be written as

$$a(x, y)u_x(x, y) + b(x, y)u_y(x, y) = c(x, y, u).$$

A first-order PDE is **quasilinear** if it can be written as

$$a(x, y, u)u_x(x, y) + b(x, y, u)u_y(x, y) = c(x, y, u).$$

Simple Case

Let c be a constant and consider

$$u_x + c\,u_y = 0$$
$$\langle u_x, u_y \rangle \cdot \langle 1, c \rangle = 0$$
$$D_{\langle 1,c \rangle} u(x,t) = 0.$$

Remarks:

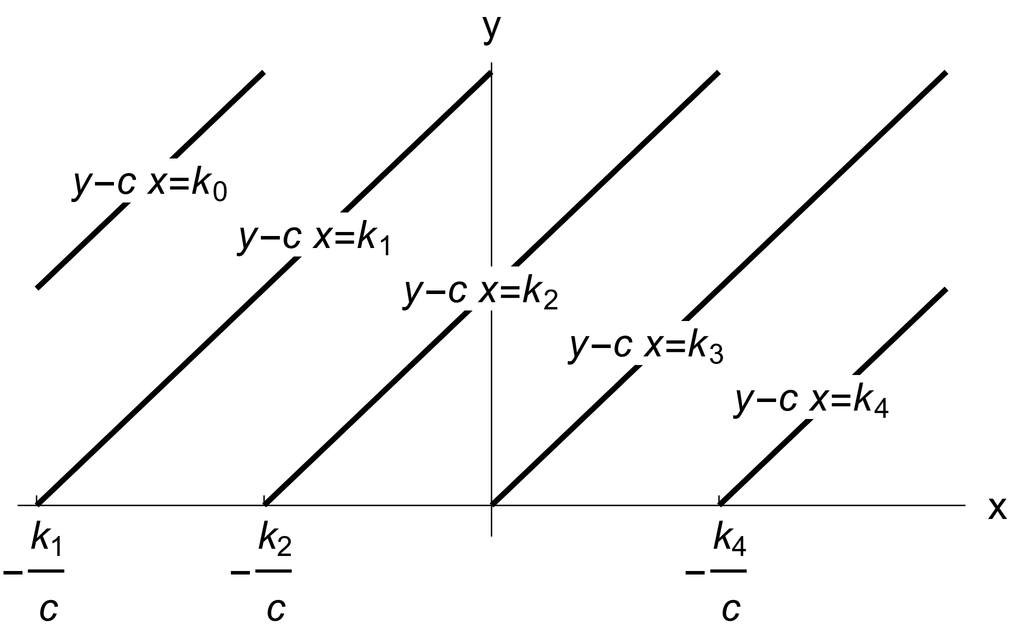
- This can be interpreted as stating the directional derivative of $u(x,y)$ in the direction of vector $\langle 1, c \rangle$ is 0.
- Function $u(x,y)$ is constant along lines of the form $y - c\,x = k$.
- Function $u(x,y) \equiv f(k)$ where f is an arbitrary differentiable function.

Illustration

The solutions of

$$u_x + c\,u_y = 0$$

must be constant along lines of the form $y - c\,x = k$.



Confirmation of Solution

Let $u(x, y) = f(y - c\,x)$ then

$$u_x = -c\,f'(y - c\,x)$$
$$u_y = f'(y - c\,x)$$

and

$$u_x + c\,u_y = -c\,f'(y - c\,x) + c\,f'(y - c\,x) = 0.$$

Functions of the form $u(x, y) = f(y - c\,x)$ are referred to as the **general solutions** to this simple PDE.

Initial Conditions

If the value of $u(x, y)$ along the line where $x = 0$ is specified, then an initial condition of the form $u(0, y) = \phi(y)$ further specifies the solution.

$$u(x, y) = f(y - c\,x) \text{ (general solution)}$$
$$u(0, y) = f(y) = \phi(y) \text{ (initial condition)}$$
$$u(x, y) = \phi(y - c\,x) \text{ (solution to IVP)}$$

General First-Order Linear PDE

The solution technique used previously can be extended to the **general first-order linear PDE** of the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y),$$

for $(x,y) \in D \subset \mathbb{R}^2$.

Vector Field (1 of 3)

Suppose $(x,y) \equiv (x(t), y(t))$ where t is a parameter, then

$$\frac{d}{dt}[u(x,y)] = u_x x'(t) + u_y y'(t).$$

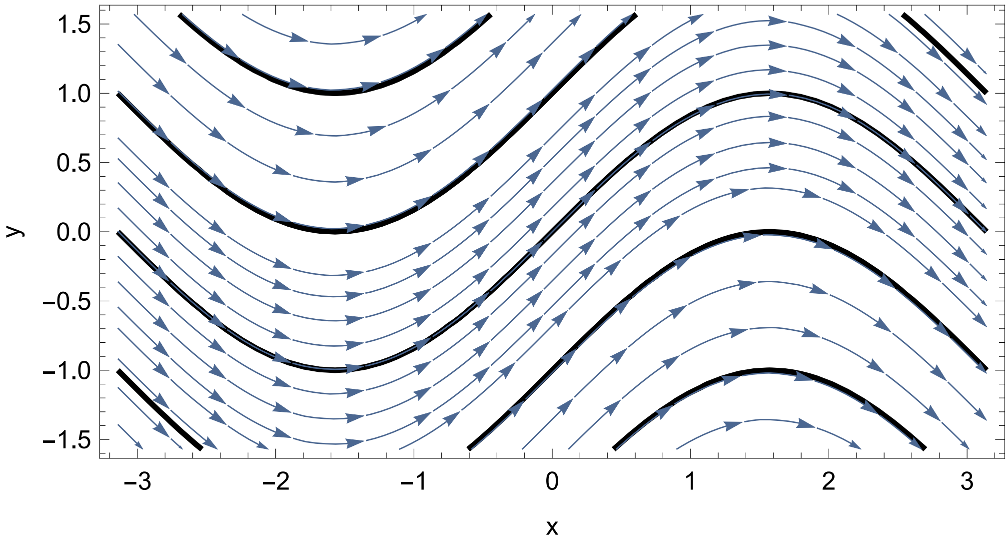
Consider

$$\begin{aligned} a(x,y)u_x + b(x,y)u_y &= c(x,y)u + d(x,y) \text{ as} \\ \frac{dx}{dt}u_x + \frac{dy}{dt}u_y &= c(x,y)u + d(x,y) \text{ where} \\ \frac{dx}{dt} &= a(x,y) \\ \frac{dy}{dt} &= b(x,y). \end{aligned}$$

Remark: the parametric curve $(x(t), y(t))$ is an **integral curve** of the **vector field** $\langle a(x,y), b(x,y) \rangle$ for all $(x,y) \in D$.

Vector Field (2 of 3)

A vector field and integral curves:



Vector Field (3 of 3)

If $\frac{dx}{dt} = a(x, y)$ and $\frac{dy}{dt} = b(x, y)$ then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b(x, y)}{a(x, y)}.$$

Suppose the implicit form of the solution of this ODE is $\phi(x, y) = k$ where k is an arbitrary constant.

Along the curves defined by $\phi(x, y) = k$ the general first-order linear PDE

$$a(x, y)u_x + b(x, y)u_y = c(x, y)u + d(x, y)$$

becomes

$$\begin{aligned} u_x + u_y \frac{b(x, y)}{a(x, y)} &= \frac{c(x, y)u + d(x, y)}{a(x, y)} \\ u_x + u_y \frac{dy}{dx} &= \frac{c(x, y)u + d(x, y)}{a(x, y)} \\ \frac{du}{dx} &= \frac{c(x, y)u + d(x, y)}{a(x, y)}. \end{aligned}$$

Ordinary Differential Equation

$$\frac{du}{dx} = \frac{c(x,y)u + d(x,y)}{a(x,y)}$$

is an ordinary differential equation for u as a function of variable x (since $y \equiv y(x)$) and arbitrary constant k.

Solving this ODE yields $u \equiv u(x,k) \equiv u(x,y)$ since $k = \phi(x,y)$.

Characteristics

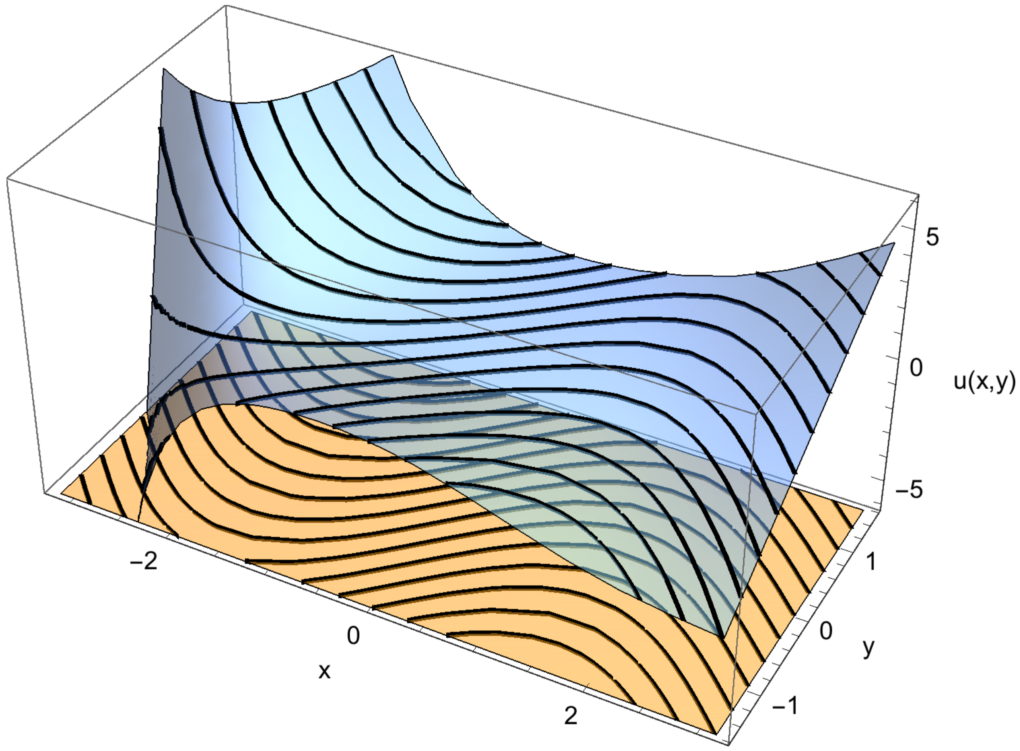
- The ordinary differential equations:

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

$$\frac{du}{dt} = c(x,y)u + d(x,y)$$

  are called the **characteristic equations**.
- The solution curves $(x(t), y(t), u(x(t), y(t)))$ to the characteristic equations are called the **characteristic curves**.
- The curves in the xy-plane of the form $(x(t), y(t))$ are called **characteristics**.

Characteristic Curves *vs.* Characteristics



Characteristics are curves in the xy-plane and characteristic curves are curves on the surface $u(x, y)$.

Example

Consider the initial value problem:

$$2u_x + 3u_y - 4u = 0$$
$$u(x, 0) = \sin x.$$

1. Find the general solution to the PDE using the method of characteristics.
2. Find the solution to the IVP.

Solution (1 of 2)

$$2u_x + 3u_y - 4u = 0$$

In this PDE we can let $a(x, y) = 2$ and $b(x, y) = 3$, thus

$$\frac{dy}{dx} = \frac{3}{2} \implies y = \frac{3}{2}x + \hat{k} \iff 2y - 3x = k.$$

Comment: the characteristics for this PDE are straight lines of the form $2y - 3x = k$.

Think of $u(x, y) = u(x, y(x))$ so that

$$\frac{du}{dx} = u_x + u_y \frac{dy}{dx} = u_x + \frac{3}{2}u_y = 2u$$

which implies

$$u(x, y) = f(k)e^{2x} = f(2y - 3x)e^{2x}$$

where f is an arbitrary differentiable function.

Solution (2 of 2)

In order to satisfy the initial condition:

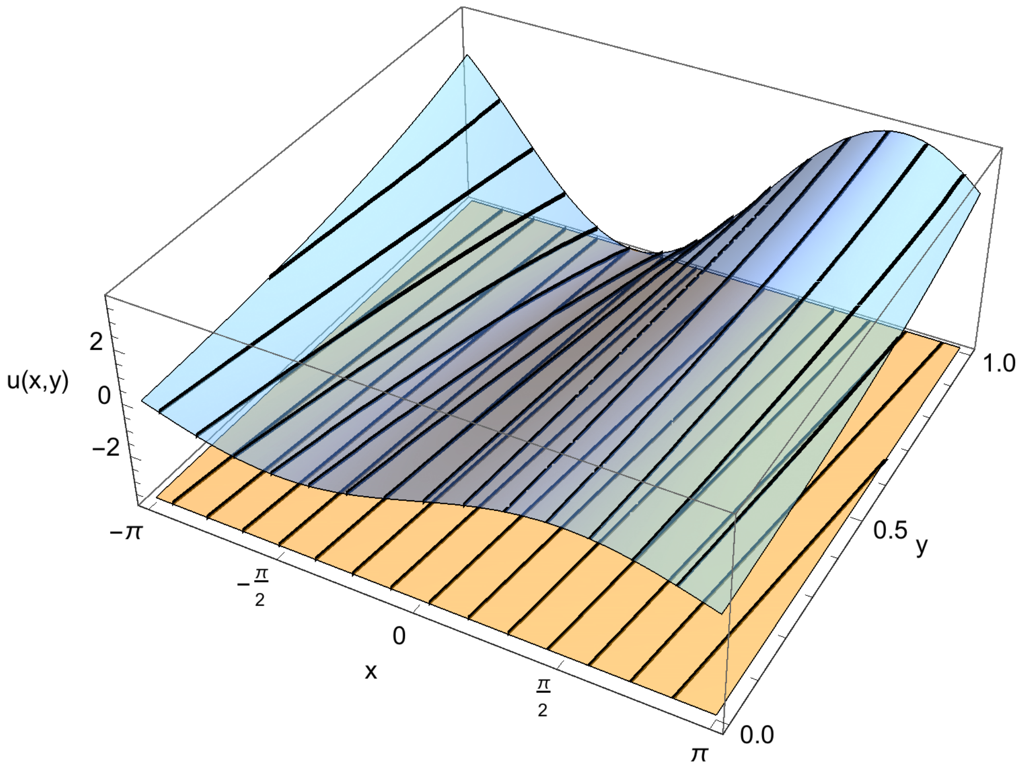
$$\begin{aligned} u(x, 0) &= \sin x \\ f(-3x)e^{2x} &= \sin x \\ f(-3x) &= e^{-2x} \sin x \\ f(z) &= -e^{2z/3} \sin(z/3). \end{aligned}$$

Thus the solution to the IVP is

$$u(x, y) = -e^{2(2y-3x)/3} \sin((2y - 3x)/3)e^{2x} = -e^{4y/3} \sin\left(\frac{2}{3}y - x\right).$$

Illustration

$$u(x, y) = -e^{4y/3} \sin\left(\frac{2}{3}y - x\right)$$



Example

Find the general solution to the first-order linear PDE:

$$-y\,u_x + x\,u_y = 0.$$

Solution (1 of 2)

If we let $a(x, y) = -y$ and $b(x, y) = x$ then the characteristics satisfy the ODE

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{x}{y} \\
y\,dy &= -x\,dx \\
\int y\,dy &= -\int x\,dx \\
\frac{1}{2}y^2 &= -\frac{1}{2}x^2 + \hat{k} \\
x^2 + y^2 &= k.
\end{aligned}$$

In this example the characteristics are circles of radius $\sqrt{k}$ for $k \geq 0$.

Solution (2 of 2)

Assuming $u(x, y) = u(x, y(x))$ then

$$\begin{aligned}
\frac{du}{dx} &= u_x + u_y \frac{dy}{dx} \\
&= u_x - \frac{x}{y} u_y \\
\frac{du}{dx} &= 0 \\
u &= f(k) \\
u(x, y) &= f(x^2 + y^2)
\end{aligned}$$

where f is an arbitrary differentiable function.

Example

Solve the initial value problem:

$$
\begin{aligned}
2x\,y\,u_x + u_y - u &= 0 \\
u(x,0) &= x,
\end{aligned}
$$

for $x > 0$ and $y > 0$.

Solution (1 of 2)

Let $a(x,y) = 2x\,y$ and $b(x,y) = 1$ then

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2x\,y} \text{ (separable ODE)} \\
y\,dy &= \frac{1}{2x}\,dx \\
\int y\,dy &= \int \frac{1}{2x}\,dx \\
\frac{1}{2}y^2 &= \frac{1}{2}\ln x + \hat{k} \\
y^2 - \ln x &= k.
\end{aligned}
$$

Thus curves of the form $y = \sqrt{k + \ln x}$ are the characteristics of the solution.

Solution (2 of 2)

To find the general solution to the PDE:

$$
\begin{aligned}
2xy\,u_x + u_y - u &= 0 \\
u_x + \frac{1}{2xy}u_y &= \frac{u}{2xy} \\
\frac{du}{dx} &= \frac{u}{2x\sqrt{k+\ln x}} \text{ (separable ODE)} \\
\frac{1}{u}\,du &= \frac{1}{2x\sqrt{k+\ln x}}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{1}{2x\sqrt{k+\ln x}}\,dx \text{ (substitute } v = k+\ln x) \\
\ln u &= \sqrt{k+\ln x} + \ln f(k) \\
u(x,y) &= f(k)e^{\sqrt{k+\ln x}} = f(y^2 - \ln x)e^{y}
\end{aligned}
$$

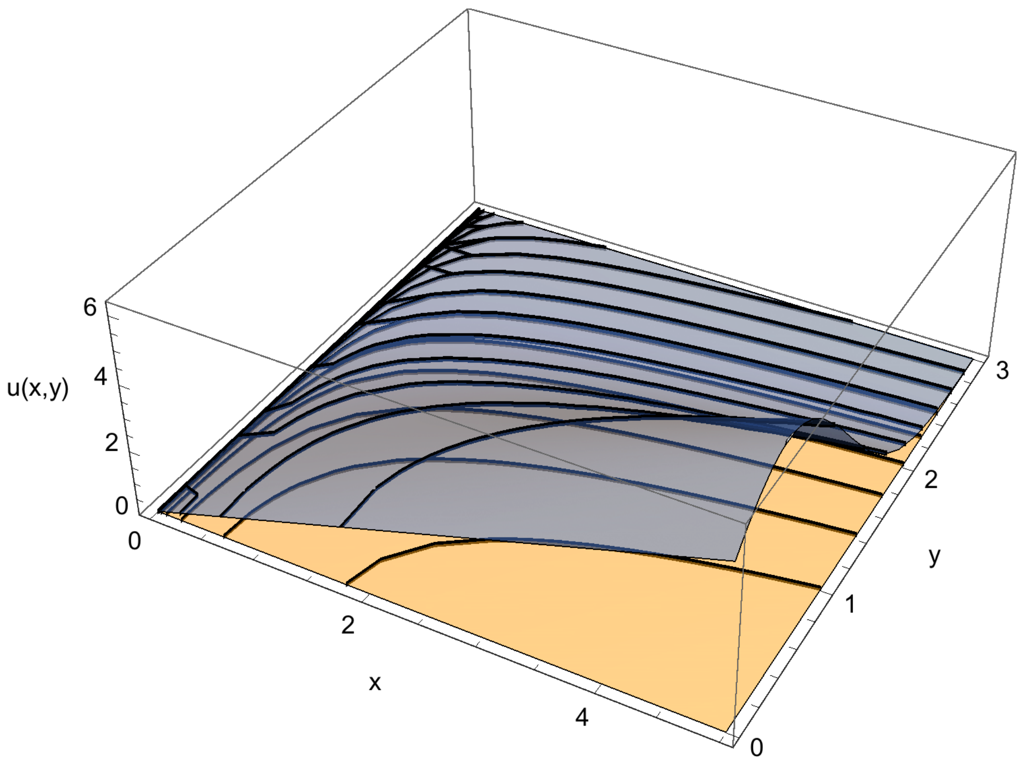
where f is an arbitrary differentiable function.

Since $u(x,0) = x$ then

$$x = f(-\ln x) \iff f(x) = e^{-x} \implies u(x,y) = e^{-y^2+\ln x}e^{y} = x\,e^{y-y^2}.$$

Illustration

$$u(x,y) = x\,e^{y-y^2}$$



Example

Show that the problem

$$
\begin{aligned}
-y\, u_x + x u_y &= 0 \\
u(x, 0) &= 3x
\end{aligned}
$$

has no solution.

Justification

From a previous example we know the characteristics are the curves of the form $x^2 + y^2 = k$ and the general solution has the form

$$u(x, y) = f(x^2 + y^2)$$

where f is an arbitrary differentiable function.

If $x \neq 0$ then

$$u(-x, 0) = -3x \neq 3x = u(x, 0)$$

but

$$u(-x, 0) = f((-x)^2) = f(x^2) = u(x, 0)$$

which is a contradiction.

Characteristics

The side condition is specified on the x-axis which intersects each characteristic curve twice (at $x = -\sqrt{k}$ and $x = \sqrt{k}$). The side condition specifies two different values of the solution, but the solution must be constant on each characteristic curve.

Example

Consider the first-order, linear PDE

$$u_x + (\cos x)u_y + u = x\,y.$$

Find the general solution to this PDE.

Solution (1 of 4)

Let $a(x, y) = 1$ and $b(x, y) = \cos x$, then the characteristics satisfy the ODE

$$\begin{aligned} \frac{dy}{dx} &= \cos x \\ dy &= \cos x\, dx \\ \int 1\, dy &= \int \cos x\, dx \\ y &= \sin x + k. \end{aligned}$$

In this example the characteristics are curves of the form $y - \sin x = k$.

Solution (2 of 4)

Assuming $u(x, y) = u(x, y(x))$ then

$$\begin{aligned} u_x + (\cos x)u_y + u &= x\, y \\ \frac{du}{dx} &= x\, y - u = x(\sin x + k) - u \\ \frac{du}{dx} + u &= x \sin x + k\, x. \end{aligned}$$

This is a first-order, linear ODE which can be solved by multiplying both sides by the integrating factor e^x and integrating.

Solution (3 of 4)

$$
\begin{aligned}
\frac{du}{dx} + u &= x \sin x + k\,x \\
\frac{d}{dx}\left[u\,e^x\right] &= x\,e^x \sin x + k\,x\,e^x \\
\int d\left[u\,e^x\right] &= \int (x\,e^x \sin x + k\,x\,e^x)\,dx \\
u\,e^x &= \frac{1}{2} e^x \left(2k(x-1) - (x-1)\cos x + x \sin x\right) + f(k) \\
u &= \frac{1}{2}\left(2k(x-1) - (x-1)\cos x + x \sin x\right) + e^{-x} f(k)
\end{aligned}
$$

where f is an arbitrary differentiable function.

Solution (4 of 4)

$$
\begin{aligned}
u &= \frac{1}{2}\left(2k(x-1) - (x-1)\cos x + x \sin x\right) + e^{-x} f(k) \\
u(x,y) &= (x-1)(y - \sin x) - \frac{x-1}{2}\cos x + \frac{x}{2}\sin x + e^{-x} f(y - \sin x)
\end{aligned}
$$

Example

Consider the first-order, linear PDE and side condition

$$y\,u_x - 4x\,u_y = 2x\,y$$
$$u(x,0) = x^4$$

Find a particular solution to this PDE which satisfies the side condition.

Solution (1 of 3)

Let $a(x,y) = y$ and $b(x,y) = -4x$, then the characteristics satisfy the ODE

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{4x}{y} \text{ (separable ODE)} \\
y\,dy &= -4x\,dx \\
\int y\,dy &= -\int 4x\,dx \\
\frac{1}{2}y^2 &= -2x^2 + \hat{k}.
\end{aligned}$$

In this example the characteristics are curves of the form $y^2 + 4x^2 = k$.

Solution (2 of 3)

Assuming $u(x, y) = u(x, y(x))$ then

$$
\begin{aligned}
y\, u_x - 4x\, u_y &= 2x\, y \\
u_x - \frac{4x}{y}\, u_y &= 2x \\
\frac{du}{dx} &= 2x \text{ (separable ODE)} \\
du &= 2x\, dx \\
\int 1\, du &= \int 2x\, dx \\
u &= x^2 + f(k) \\
u(x, y) &= x^2 + f(4x^2 + y^2)
\end{aligned}
$$

where f is an arbitrary differentiable function.

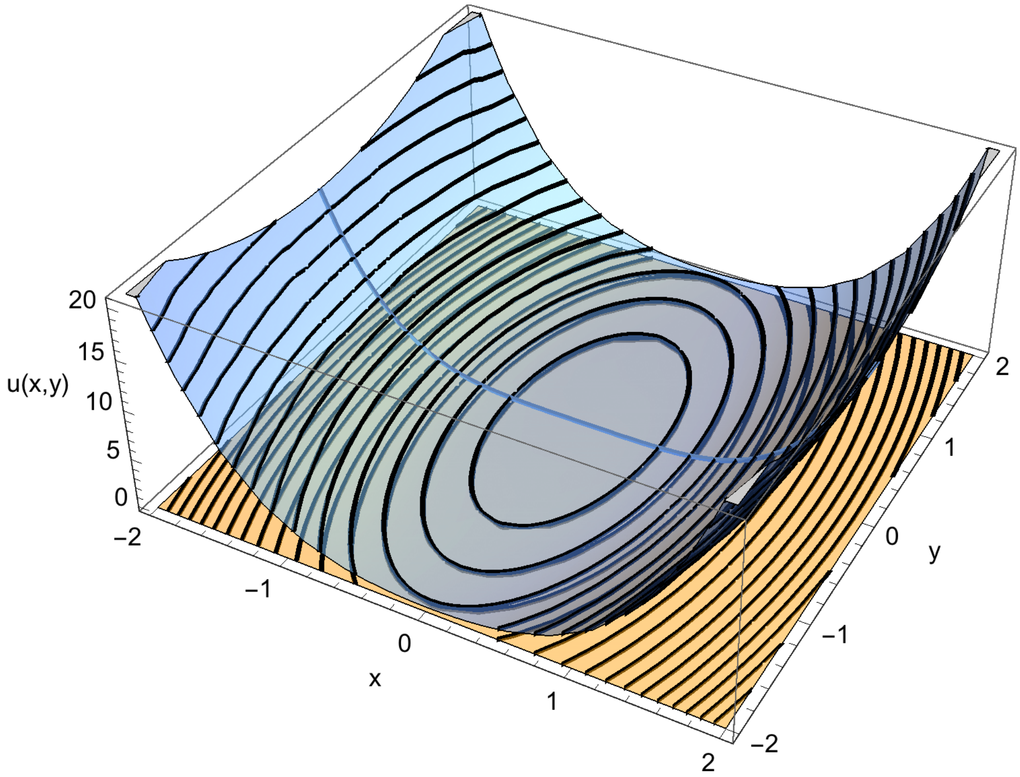
Solution (3 of 3)

To satisfy the side condition we need

$$
\begin{aligned}
u(x, 0) &= x^2 + f(4x^2) \\
x^4 &= x^2 + f(4x^2) \\
f(4x^2) &= x^4 - x^2 \\
f(z) &= \frac{z^2}{16} - \frac{z}{4} \\
u(x, y) &= x^2 + \frac{(4x^2 + y^2)^2}{16} - \frac{4x^2 + y^2}{4}
\end{aligned}
$$

Illustration

$$u(x,y) = x^2 + \frac{(4x^2+y^2)^2}{16} - \frac{4x^2+y^2}{4}$$



Example

Consider the first-order, linear PDE and side condition

$$y\,u_x - 4x\,u_y = 2x\,y$$
$$u(x,0) = x^3$$

Show there is no solution to this PDE which satisfies the side condition.

Justification

From the work done previously we know the general solution to the PDE takes the form

$$u(x, y) = x^2 + f(4x^2 + y^2)$$

where f is an arbitrary differentiable function.

Suppose $u(x, y)$ is such that $u(x, 0) = x^3$, then for any x

$$u(-x, 0) = (-x)^2 + f(4(-x)^2 + 0^2) = x^2 + f(4x^2 + 0^2) = u(x, 0)$$

However, when $x \neq 0$,

$$u(-x, 0) = (-x)^3 \neq x^3 = u(x, 0)$$

which is a contradiction.

Homework

- Read Section 2.1
- Exercises: 1–5