The Gibbs Phenomenon

MATH 365 *Partial Differential Equations*

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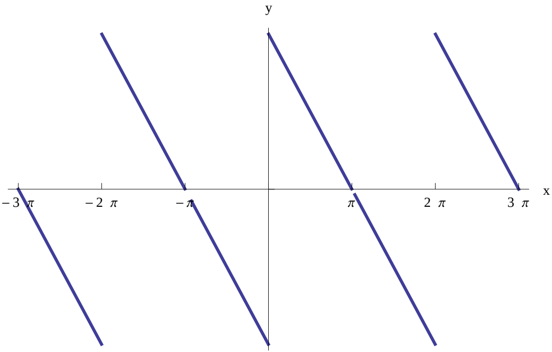
Periodic Function with Discontinuity

Let

$$f(x) = \begin{cases} -\frac{1}{2}(\pi + x) & \text{if } -\pi \le x < 0, \\ \frac{1}{2}(\pi - x) & \text{if } 0 \le x < \pi. \end{cases}$$

Let $F(x)$ be the 2π-periodic extension of $f(x)$ to $\mathbb{R}$.



Fourier Series

Since $F(x)$ is an odd function we can represent it by a **Fourier Sine Series**.

$$F(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} \frac{1}{2}(\pi - x) \sin(nx)\, dx = \frac{1}{n}$$

for $n = 1, 2, \ldots$.

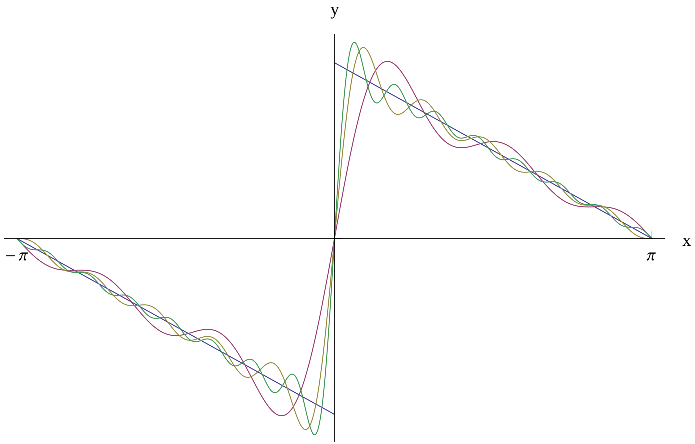
$$F(x) \sim \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx)$$

Partial Sums

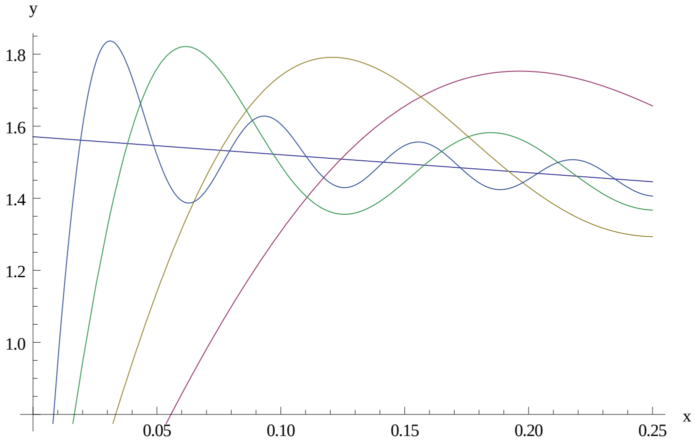
Let

$$S_N(x) = \sum_{n=1}^{N} \frac{1}{n} \sin(nx)$$

and consider the convergence of $S_N(x)$ to $F(x)$.



Zoom In Near $x = 0$



It appears that there is a non-decreasing deviation of $S_N(x)$ from $F(x)$ for x near 0 even as $N \to \infty$.

Deviation

Let $x = \pi/(N+1)$ then

$$S_N\left(\frac{\pi}{N+1}\right) = \sum_{n=1}^{N} \frac{1}{n} \sin \frac{n\pi}{N+1}.$$

Remark: this expression can be interpreted as a Riemann sum approximation to $\int_0^{\pi} \frac{\sin x}{x}\, dx$.

Let

$$\begin{aligned} \Delta x &= \frac{\pi}{N+1} \\ x_n &= \frac{n\pi}{N+1} \quad \text{for } n = 0, 1, \ldots, N+1 \\ w_n &= \frac{n\pi}{N+1} \quad \text{for } n = 1, 2, \ldots, N+1. \end{aligned}$$

Riemann Sum

$$\begin{aligned}
\int_0^{\pi} \frac{\sin x}{x}\, dx &\approx \sum_{n=1}^{N+1} \frac{\sin w_n}{w_n} \Delta x \\
&= \sum_{n=1}^{N+1} \frac{\sin \frac{n\pi}{N+1}}{\frac{n\pi}{N+1}} \frac{\pi}{N+1} \\
&= \sum_{n=1}^{N+1} \frac{1}{n} \sin \frac{n\pi}{N+1} \\
&= \sum_{n=1}^{N} \frac{1}{n} \sin \frac{n\pi}{N+1},
\end{aligned}$$

since $\sin \dfrac{(N+1)\pi}{N+1} = 0$. Thus

$$\int_0^{\pi} \frac{\sin x}{x}\, dx \approx S_N\left(\frac{\pi}{N+1}\right).$$

Limit

Since

$$\int_0^{\pi} \frac{\sin x}{x}\, dx \approx S_N\left(\frac{\pi}{N+1}\right),$$

then

$$\lim_{N\to\infty} S_N\left(\frac{\pi}{N+1}\right) = \int_0^{\pi} \frac{\sin x}{x}\, dx.$$

We need to evaluate the definite integral.
Recall the Taylor series for $\sin x$.

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \Longrightarrow \quad \frac{\sin x}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}$$

Definite Integral

$$\begin{aligned}
\int_0^{\pi} \frac{\sin x}{x}\, dx &= \int_0^{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}\, dx \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^{\pi} x^{2k}\, dx \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{(2k+1)(2k+1)!}
\end{aligned}$$

Remark: The definite integral is expressed as an alternating series. We can use the Alternating Series Test to confirm its convergence and to estimate the value of the definite integral.

Alternating Series Test

Theorem (Alternating Series Test)

The alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ *is convergent if*

1. $0 < a_{k+1} \leq a_k$ *for all* $k \geq 1$*, and*
2. $\lim_{k\to\infty} a_k = 0$*.*

Theorem

Suppose $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ *is a convergent alternating series. If* S *is the sum of the series and* S_n *is the* n^{th} *partial sum of the series, then*

$$|S - S_n| \leq a_{n+1}$$

for all n*.*

Estimation of the Definite Integral

Let $S = \int_0^{\pi} \frac{\sin x}{x}\, dx$ then

$$S \approx T_M = \sum_{k=0}^{M} \frac{(-1)^k \pi^{2k+1}}{(2k+1)(2k+1)!}$$

and

$$|S - T_M| \leq \left| \frac{(-1)^{M+1} \pi^{2M+3}}{(2M+3)(2M+3)!} \right| = \frac{\pi^{2M+3}}{(2M+3)(2M+3)!}.$$

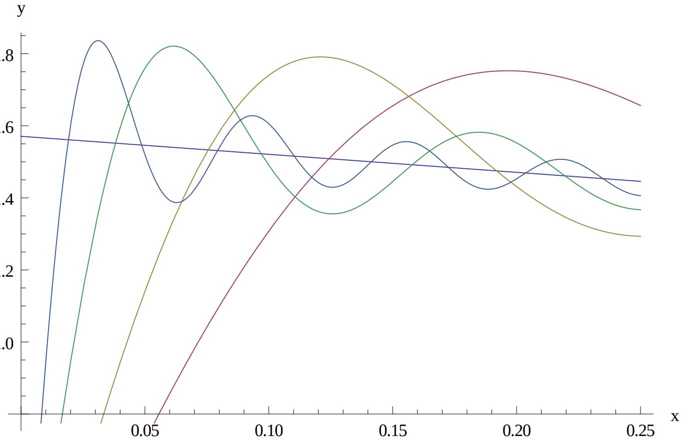
When $M = 15$ we have

$$|S - T_M| \leq \frac{\pi^{33}}{(33)33!} \approx 8.88682 \times 10^{-23},$$

thus

$$S \approx T_{15} \approx 1.85194.$$

Deviation Revisited (1 of 2)



$$F\left(\frac{\pi}{N+1}\right) - S_N\left(\frac{\pi}{N+1}\right) = \frac{N\pi}{2(N+1)} - \sum_{n=1}^{N} \frac{1}{n} \sin \frac{n\pi}{N+1}$$

Deviation Revisited (2 of 2)

$$
\begin{aligned}
&\lim_{N\to\infty}\left|F\left(\frac{\pi}{N+1}\right) - S_N\left(\frac{\pi}{N+1}\right)\right| \\
&= \left|\lim_{N\to\infty}\left(\frac{N\pi}{2(N+1)} - \sum_{n=1}^{N}\frac{1}{n}\sin\frac{n\pi}{N+1}\right)\right| \\
&= \left|\frac{\pi}{2} - \int_0^{\pi}\frac{\sin x}{x}\,dx\right| \\
&\approx \left|\frac{\pi}{2} - 1.85194\right| \\
&\approx 0.281141
\end{aligned}
$$

Thus there is always a deviation of at least 0.28 for all N even as $N \to \infty$.