Heat Equation on a Rectangular Domain

Partial Differential Equations

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Objectives

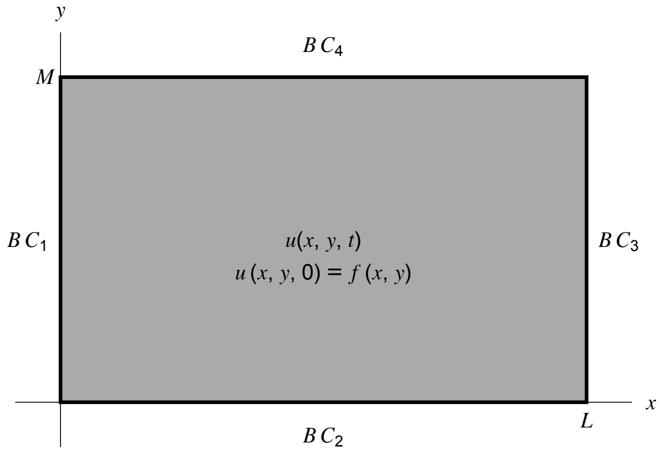
In this lesson we will learn:

- to find the eigenfunctions and eigenvalues for common boundary value problems on a rectangular domain,
- to use a double Fourier series to express the solution to an initial, boundary value problem for the heat equation on a rectangular domain.

Geometry of the Domain

Consider the two-dimensional heat equation and its initial conditions on the rectangle below.

$$u_t = \kappa(u_{xx} + u_{yy}) \text{ for } (x, y) \in (0, L) \times (0, M),\ t > 0$$
$$u(x, y, 0) = f(x, y) \text{ for } (x, y) \in [0, L] \times [0, M].$$



Separation of Variables

The method of separation of variables will be used to find the eigenfunctions and eigenvalues corresponding to the boundary conditions imposed on the rectangle.

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Suppose $u(x, y, t) = X(x)Y(y)T(t)$ then

$$X(x)Y(y)T'(t) = \kappa(X''(x)Y(y)T(t) + X(x)Y''(y)T(t)) \text{ or}$$
$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \alpha$$

The expression α must be a constant (why?).

Time-independent Portion of Solution

$$\begin{aligned}\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} &= \alpha \\ \frac{X''(x)}{X(x)} &= \alpha - \frac{Y''(y)}{Y(y)} = \beta.\end{aligned}$$

- β must be a constant.
- The constants α and β are further restricted by the boundary conditions imposed at the edges of rectangle R.

Dirichlet Boundary Conditions

Suppose the boundary conditions on all four sides of the rectangle are of homogeneous Dirichlet type.

$$
\begin{aligned}
u(0, y, t) &= u(L, y, t) = 0 \\
u(x, 0, t) &= u(x, M, t) = 0
\end{aligned}
$$

for $0 \leq y \leq M$, $0 \leq x \leq L$, and $t \geq 0$.

The nonzero solutions to the boundary value problem,

$$
\begin{aligned}
X''(x) - \beta X(x) &= 0 \\
X(0) = X(L) &= 0
\end{aligned}
$$

exist only when $\beta = -\sigma_n^2$ where $\sigma_n = n\pi/L$, $n \in \mathbb{N}$. The nontrivial solutions corresponding to $-\sigma_n^2$ are nonzero constant multiples of

$$
X_n(x) = \sin\left(\frac{n\pi x}{L}\right).
$$

Finding the y-dependent Factor

$$\alpha - \frac{Y''(y)}{Y(y)} = \beta = -\left(\frac{n\pi}{L}\right)^2$$

$$Y''(y) - \left[\alpha + \left(\frac{n\pi}{L}\right)^2\right] Y(y) = 0$$

Nonzero solutions satisfying the homogeneous Dirichlet boundary conditions exist when

$$\alpha = -\pi^2 \left[\left(\frac{m}{M}\right)^2 + \left(\frac{n}{L}\right)^2\right]$$

and are nonzero constant multiples of

$$Y_m(y) = \sin\left(\frac{m\pi y}{M}\right)$$

for $m = 1, 2, \ldots$ and $n = 1, 2, \ldots$.

Summary of Product Solution So Far

The functions

$$X_n(x)Y_m(y) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{M}\right)$$

will be called the eigenfunctions of the boundary value problem and the constants

$$\lambda_{m,n} = \pi^2\left[\left(\frac{m}{M}\right)^2 + \left(\frac{n}{L}\right)^2\right]$$

for $m, n \in \mathbb{N}$ will be their corresponding eigenvalues.

Finding the t-dependent Factor

$$\frac{T'(t)}{\kappa T(t)} = \alpha$$

$$T'(t) = -\kappa\pi^2 \left[\left(\frac{m}{M}\right)^2 + \left(\frac{n}{L}\right)^2 \right] T(t)$$

$$T_{m,n}(t) = e^{-\kappa\pi^2\left[\left(\frac{m}{M}\right)^2+\left(\frac{n}{L}\right)^2\right]t}$$

for $m, n \in \mathbb{N}$.

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for $m, n \in \mathbb{N}$.

The product solutions take the form,

$$u_{m,n}(x, y, t) = e^{-\kappa\pi^2 \left[\left(\frac{m}{M}\right)^2 + \left(\frac{n}{L}\right)^2 \right] t} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{M}\right).$$

Double Fourier Series

Using the principle of superposition, the general solution to this initial boundary value problem can be expressed as

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} e^{-\kappa\pi^2 \left[\left(\frac{m}{M}\right)^2 + \left(\frac{n}{L}\right)^2\right] t} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{M}\right),$$

where the coefficients $a_{m,n}$ must be selected so that the solution satisfies the initial condition.

Satisfying the Initial Conditions

$$u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{M}\right) = f(x, y)$$

Since the eigenfunctions are orthogonal on R, then

$$a_{m,n} = \frac{4}{LM} \int_0^M \int_0^L f(x, y) \sin\left(\frac{m\pi y}{M}\right) \sin\left(\frac{n\pi x}{L}\right)\, dx\, dy,$$

provided the integrals exist.

Example

Find the solution to the initial boundary value problem:

$$
\begin{aligned}
u_t &= u_{xx} + u_{yy} \text{ for } (x, y) \in (0, 1) \times (0, 2) \text{ and } t > 0 \\
u(0, y, t) &= u(1, y, t) = 0 \text{ for } 0 < y < 2 \text{ and } t > 0 \\
u(x, 0, t) &= u(x, 2, t) = 0 \text{ for } 0 < x < 1 \text{ and } t > 0 \\
u(x, y, 0) &= 1000x(1 - x) \sin(2\pi y) \text{ for } (x, y) \in [0, 1] \times [0, 2].
\end{aligned}
$$

Solution (1 of 2)

In this example $\kappa = 1$, $L = 1$, and $M = 2$ and the boundary conditions are of the homogeneous Dirichlet type.

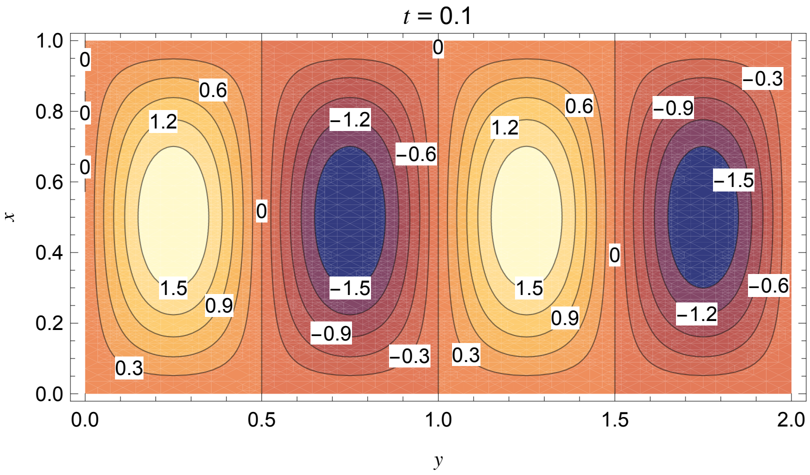
$$a_{m,n} = \frac{4}{(1)(2)} \int_0^2 \int_0^1 1000x(1-x)\sin(2\pi y)\sin(n\pi x)\sin\left(\frac{m\pi y}{2}\right)\,dx\,dy$$

By the orthogonality property of the sine functions $a_{m,n} = 0$ for $m \neq 4$.

$$\begin{aligned}
a_{4,n} &= 2000 \int_0^2 \int_0^1 x(1-x)\sin^2(2\pi y)\sin(n\pi x)\,dx\,dy \\
&= 2000 \int_0^1 x(1-x)\sin(n\pi x)\left[\int_0^2 \sin^2(2\pi y)\,dy\right]\,dx \\
&= 2000 \int_0^1 x(1-x)\sin(n\pi x)\,dx \\
&= \begin{cases} \frac{8000}{n^3\pi^3} & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}
\end{aligned}$$

Solution (2 of 2)

$$u(x, y, t) \sim \frac{8000}{\pi^3} \sin(2\pi y) \sum_{n=1}^{\infty} \frac{e^{-\pi^2((2n-1)^2+4)t}}{(2n-1)^3} \sin((2n-1)\pi x).$$



Mixed Dirichlet/Neumann Boundary Conditions

Suppose the boundary conditions are of homogeneous Dirichlet type on a set of opposing boundaries and of homogeneous Neumann type on the other set of opposing boundaries.

$$u(0, y, t) = u(L, y, t) = 0$$
$$u_y(x, 0, t) = u_y(x, M, t) = 0$$

for $0 \le y \le M$, $0 \le x \le L$, and $t \ge 0$.

The nonzero solutions to the boundary value problem,

$$\begin{aligned} X''(x) - \beta X(x) &= 0 \\ X(0) = X(L) &= 0 \end{aligned}$$

exist only when $\beta = -\sigma_n^2$ where $\sigma_n = n\pi/L$, $n \in \mathbb{N}$. The nontrivial solutions corresponding to $-\sigma_n^2$ are nonzero constant multiples of

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

Finding the y-dependent Factor

$$\alpha - \frac{Y''(y)}{Y(y)} = \beta = -\left(\frac{n\pi}{L}\right)^2$$

$$Y''(y) - \left[\alpha + \left(\frac{n\pi}{L}\right)^2\right] Y(y) = 0$$

Nonzero solutions satisfying the homogeneous Neumann boundary conditions exist when

$$\alpha = -\pi^2 \left[\left(\frac{m}{M}\right)^2 + \left(\frac{n}{L}\right)^2\right]$$

and are nonzero constant multiples of

$$Y_m(y) = \cos\left(\frac{m\pi y}{M}\right)$$

for $m = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$.

Summary of Product Solution So Far

The functions

$$X_n(x)Y_m(y) = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{M}\right)$$

will be called the eigenfunctions of the boundary value problem and the constants

$$\lambda_{m,n} = \pi^2\left[\left(\frac{m}{M}\right)^2 + \left(\frac{n}{L}\right)^2\right]$$

for $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ will be their corresponding eigenvalues.

Finding the t-dependent Factor

$$\frac{T'(t)}{\kappa T(t)} = \alpha$$

$$T'(t) = -\kappa\pi^2 \left[\left(\frac{m}{M}\right)^2 + \left(\frac{n}{L}\right)^2 \right] T(t)$$

$$T_{m,n}(t) = e^{-\kappa\pi^2 \left[\left(\frac{m}{M}\right)^2 + \left(\frac{n}{L}\right)^2 \right] t}$$

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for $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$.

The product solutions take the form,

$$u_{m,n}(x, y, t) = e^{-\kappa\pi^2 \left[\left(\frac{m}{M}\right)^2 + \left(\frac{n}{L}\right)^2 \right] t} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{M}\right).$$

Double Fourier Series

Using the principle of superposition, the general solution to this initial boundary value problem can be expressed as

$$u(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{m,n} e^{-\kappa \pi^2 \left[\left(\frac{m}{M} \right)^2 + \left(\frac{n}{L} \right)^2 \right] t} \sin \left(\frac{n \pi x}{L} \right) \cos \left(\frac{m \pi y}{M} \right),$$

where the coefficients $a_{m,n}$ must be selected so that the solution satisfies the initial condition.

Satisfying the Initial Conditions

$$u(x, y, 0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{M}\right) = f(x, y)$$

Since the eigenfunctions are orthogonal on R, then

$$a_{m,n} = \frac{4}{LM} \int_0^M \int_0^L f(x, y) \cos\left(\frac{m\pi y}{M}\right) \sin\left(\frac{n\pi x}{L}\right) \, dx \, dy,$$

provided the integrals exist.

Example

Find the solution to the initial boundary value problem:

$$\begin{aligned} u_t &= u_{xx} + u_{yy} \text{ for } (x, y) \in (0, 1) \times (0, 1) \text{ and } t > 0 \\ u(0, y, t) &= 0 \text{ and } u(1, y, t) = 0 \text{ for } 0 < y < 1 \text{ and } t > 0 \\ u_y(x, 0, t) &= u_y(x, 1, t) = 0 \text{ for } 0 < x < 1 \text{ and } t > 0 \\ u(x, y, 0) &= (1 + 4x) + \sin(\pi x)\cos(2\pi y) \text{ for } (x, y) \in [0, 1] \times [0, 1]. \end{aligned}$$

Solution (1 of 2)

In this example $\kappa = 1$, $L = 1$, and $M = 1$.

The double series solution for the initial boundary value problem will resemble

$$u(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{m,n} e^{-\pi^2 (m^2+n^2)t} \sin(n\pi x) \cos(m\pi y).$$

Solution (2 of 2)

When $t = 0$ and $m = 0$,

$$a_{0,n} = 2 \int_0^1 \int_0^1 \sin(\pi x) \cos(2\pi y) \sin(n\pi x)\, dy\, dx = 0.$$

Solution (2 of 2)

When $t = 0$ and $m = 0$,

$$a_{0,n} = 2 \int_0^1 \int_0^1 \sin(\pi x) \cos(2\pi y) \sin(n\pi x) \, dy \, dx = 0.$$

When $t = 0$ and $m \in \mathbb{N}$, then

$$a_{m,n} = 4 \int_0^1 \int_0^1 [\sin(\pi x) \cos(2\pi y)] \cos(m\pi y) \sin(n\pi x) \, dy \, dx = 0.$$

except for the case where $n = 1$ and $m = 2$.

$$a_{2,1} = 4 \int_0^1 \int_0^1 \sin^2(\pi x) \cos^2(2\pi y) \, dy \, dx = 1$$

The solution can be written concisely as

$$u(x, y, t) = e^{-5\pi^2 t} \sin(\pi x) \cos(2\pi y).$$

Solution (3 of 3)