Steady-State Temperature in a Sphere

Partial Differential Equations

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Objectives

In this lesson we will learn:

- to solve Laplace's equation in spherical coordinates, and
- describe the steady-state temperature distribution in a sphere.

Laplace's Equation in Spherical Coordinates

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\left(\frac{\partial^2 u}{\partial \varphi^2} + \cot\varphi \frac{\partial u}{\partial \varphi} + \csc^2\varphi \frac{\partial^2 u}{\partial \theta^2}\right) = 0$$

The solutions to this equation should be

- bounded as $\rho \to 0^+$,
- 2π-periodic in variable θ, and
- bounded as $\varphi \to 0^+$ and $\varphi \to \pi^-$.

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Assuming a product solution of the form $u(\rho, \varphi, \theta) = R(\rho)\Theta(\theta)\Phi(\varphi)$ we have shown that $\Theta(\theta)\Phi(\varphi) = Y_n^m(\varphi, \theta)$ for $n = 0, 1, 2, \ldots$ and $m = -n, -n+1, \ldots, n$, the spherical harmonic functions.

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The radially dependent factor of the solution must satisfy the ordinary differential equation,

$$\rho^2 \frac{R''(\rho)}{R(\rho)} + 2\rho \frac{R'(\rho)}{R(\rho)} = n(n+1)$$

$$\rho^2 R''(\rho) + 2\rho R'(\rho) - n(n+1)R(\rho) = 0.$$

This is Euler's equation which has general solution $R_n(\rho) = a_n \rho^n + b_n \rho^{-1-n}$.

Solutions Inside/Outside a Sphere

Solutions to Laplace's equation within a sphere are product solutions of the form

$$u_{n,m}(\rho, \varphi, \theta) = a_{n,m} \rho^n Y_n^m(\varphi, \theta)$$

while solutions outside a sphere (which should be bounded as $\rho \to \infty$ take the form

$$v_{n,m}(\rho, \varphi, \theta) = b_{n,m} \rho^{-1-n} Y_n^m(\varphi, \theta).$$

The superposition principle permits a sum of these product solutions to solve Laplaces's equation and the orthogonality properties of the factors coupled with a boundary condition allow the coefficients $a_{n,m}$ or $b_{n,m}$ to be determined.

Example

Consider a solid spherical shell of radius 1 whose center lies at the origin of a three-dimensional coordinate system and for which the temperature of the surface of the northern hemisphere is kept at 100C and the temperature of the surface of the southern hemisphere is 0C. Find an expression for the steady-state temperature distribution within the sphere.

Solution (1 of 4)

This example can be summarized as finding the solution to the following boundary value problem.

$$\Delta u = 0 \text{ for } 0 < \rho < 1,\, 0 \le \varphi \le \pi,\, 0 \le \theta < 2\pi$$

$$u(1, \varphi, \theta) = \begin{cases} 100 & \text{for } 0 \le \varphi \le \frac{\pi}{2},\, 0 \le \theta < 2\pi \\ 0 & \text{for } \frac{\pi}{2} < \varphi \le \pi,\, 0 \le \theta < 2\pi \end{cases}$$

Since the boundary condition is invariant in the azimuthal angle θ, the solution to Laplace's equation should not depend on θ. This implies $m = 0$ in the spherical harmonic functions or equivalently that

$$u(\rho, \varphi, \theta) = u(\rho, \varphi) = \sum_{n=0}^{\infty} a_n \rho^n Y_n^0(\varphi, \theta) = \sum_{n=0}^{\infty} a_n \rho^n P_n(cos\varphi)$$

where $P_n(x)$ is the nth Legendre polynomial.

Solution (2 of 4)

The boundary condition $u(1, \varphi)$ determines the values of the series coefficients a_n,

$$u(1, \varphi) = \sum_{n=0}^{\infty} a_n P_n(\cos \varphi) = \begin{cases} 100 & \text{for } 0 \leq \varphi \leq \pi/2, \\ 0 & \text{for } \pi/2 < \varphi \leq \pi. \end{cases}$$

Multiply both sides of the equation by $P_m(\cos \varphi) \sin \varphi$ and integrate from from $\varphi = 0$ to $\varphi = \pi$.

$$\sum_{n=0}^{\infty} a_n \int_0^{\pi} P_n(\cos \varphi) P_m(\cos \varphi) \sin \varphi \, d\varphi = 100 \int_0^{\pi/2} P_m(\cos \varphi) \sin \varphi \, d\varphi$$

$$\sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x) P_m(x) \, dx = 100 \int_0^{1} P_m(x) \, dx$$

Solution (3 of 4)

$$\sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x) P_m(x)\, dx = 100 \int_0^1 P_m(x)\, dx$$

The only nonzero term in the infinite series on the left-hand side of the equation occurs when $m = n$, thus

$$\frac{2a_n}{2n+1} = 100 \int_0^1 P_n(x)\, dx.$$

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Evaluating the definite integral reveals $a_0 = 50$, $a_{2n} = 0$ for $n \in \mathbb{N}$, and

$$a_{2n-1} = \frac{50(-1)^n (4n-1)(2n)!}{2^{2n}(n!)^2(2n-1)} \text{ for } n \in \mathbb{N}.$$

The formal solution describing the temperature distribution on the sphere is

$$u(\rho, \varphi) = 50 - 50 \sum_{n=1}^{\infty} \frac{(-1)^n (2n)!(4n-1)}{2^{2n}(2n-1)(n!)^2} \rho^{2n-1} P_{2n-1}(\cos \varphi).$$

Solution (4 of 4)