Vibrating Sphere

Partial Differential Equations

J Robert Buchanan

Department of Mathematics

Fall 2025

Objectives

In this lesson we will:

- explore the mathematical modeling describing the displacement of a circular membrane from equilibrium, and
- use separation of variables and the spherical coordinate system to determine a series solution to the initial boundary value problem.

Model of a Vibrating Sphere

- Consider a flexible membrane in the shape of a hollow sphere of radius ρ_0.
- If a point on the surface of the sphere is displaced radially from the equilibrium radius ρ_0, the tension in the surrounding area will attempt to restore the point to the equilibrium radius.

Let $u(\varphi, \theta, t)$ be the radial displacement of the point $(\rho_0, \varphi, \theta)$ from equilibrium at time t.

$$
\begin{aligned}
u_{tt} &= c^2 \Delta u - \omega^2 u \text{ for } 0 < \varphi < \pi,\ -\pi < \theta < \pi,\ t > 0 \\
u(\varphi, \theta, 0) &= f(\varphi, \theta) \text{ for } 0 < \varphi < \pi,\ -\pi < \theta < \pi \\
u_t(\varphi, \theta, 0) &= g(\varphi, \theta) \text{ for } 0 < \varphi < \pi,\ -\pi < \theta < \pi
\end{aligned}
$$

Separation of Variables (1 of 2)

If $u(\varphi, \theta, t) = F(\varphi, \theta)T(t)$ is a product solution to the partial differential equation, then

$$\begin{aligned} F(\varphi, \theta)T''(t) &= c^2 T(t) \left(\Delta F(\varphi, \theta)\right) - \omega^2 F(\varphi, \theta)T(t) \\ \frac{1}{c^2}\left(\frac{T''(t)}{T(t)} + \omega^2\right) &= \frac{\Delta F(\varphi, \theta)}{F(\varphi, \theta)}. \end{aligned}$$

The left-hand side of the equation depends only on t and the right-hand side depends only on φ and θ, thus both sides of the equation are equal to a constant $-\lambda$.

Separation of Variables (1 of 2)

If $u(\varphi, \theta, t) = F(\varphi, \theta)T(t)$ is a product solution to the partial differential equation, then

$$F(\varphi, \theta)T''(t) = c^2 T(t)\left(\Delta F(\varphi, \theta)\right) - \omega^2 F(\varphi, \theta)T(t)$$
$$\frac{1}{c^2}\left(\frac{T''(t)}{T(t)} + \omega^2\right) = \frac{\Delta F(\varphi, \theta)}{F(\varphi, \theta)}.$$

The left-hand side of the equation depends only on t and the right-hand side depends only on φ and θ, thus both sides of the equation are equal to a constant $-\lambda$.

The right-hand side of the equation implies the differential equation

$$\Delta F(\varphi, \theta) + \lambda F(\varphi, \theta) = 0$$

The eigenvalues are $\lambda \equiv \lambda_n = n(n+1)$ for $n = 0, 1, 2, \ldots$ and the eigenfunctions are the spherical harmonics $Y_n^m(\varphi, \theta)$ for $m = -n, -n+1, \ldots, n$.

Separation of Variables (2 of 2)

The t-dependent portion of the product solution satisfies the ordinary differential equation,

$$T''(t) + (\omega^2 + c^2 n(n+1))T(t) = 0$$

which has the general solution,

$$T_n(t) = c_1 \cos\left(\sqrt{\omega^2 + c^2 n(n+1)}t\right) + c_2 \sin\left(\sqrt{\omega^2 + c^2 n(n+1)}t\right).$$

Thus a product solution has the form

$$u_{n,m}(\varphi, \theta, t) = Y_n^m(\varphi, \theta)\left(a_{n,m}\cos(\sqrt{\omega^2 + c^2\lambda_n}t) + b_{n,m}\sin(\sqrt{\omega^2 + c^2\lambda_n}t)\right)$$

for $n = 0, 1, 2, \ldots$ and $m = -n, -n+1, \ldots, n$.

Series Solution (1 of 2)

The series solution to the initial boundary value problem has the form

$$u(\varphi, \theta, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Y_n^m(\varphi, \theta) \left(a_{n,m} \cos(\sqrt{\omega^2 + c^2 \lambda_n} t) + b_{n,m} \sin(\sqrt{\omega^2 + c^2 \lambda_n} t) \right).$$

Series Solution (1 of 2)

The series solution to the initial boundary value problem has the form

$$u(\varphi, \theta, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Y_n^m(\varphi, \theta) \left(a_{n,m} \cos(\sqrt{\omega^2 + c^2 \lambda_n} t) + b_{n,m} \sin(\sqrt{\omega^2 + c^2 \lambda_n} t) \right).$$

If $t = 0$,

$$u(\varphi, \theta, 0) = f(\varphi, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m} Y_n^m(\varphi, \theta).$$

Multiply both sides by the complex conjugate of $Y_{\hat{n}}^{\hat{m}}(\varphi, \theta) \sin \varphi$ and integrate with respect to θ over interval $[-\pi, \pi]$ and with respect to φ over $[0, \pi]$. The only nonzero term in the infinite series occurs when $m = \hat{m}$ and $n = \hat{n}$. In this case,

$$\begin{aligned} a_{n,m} &= \int_{-\pi}^{\pi} \int_0^{\pi} f(\varphi, \theta) \overline{Y_n^m(\varphi, \theta)} \sin \varphi \, d\varphi \, d\theta \\ &= \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} \int_{-\pi}^{\pi} \int_0^{\pi} f(\varphi, \theta) e^{-im\theta} P_n^m(\cos \varphi) \sin \varphi \, d\varphi \, d\theta. \end{aligned}$$

Series Solution (2 of 2)

Differentiate the solution with respect to t and set $t = 0$.

$$u_t(\varphi, \theta, 0) = g(\varphi, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{n,m} \sqrt{\omega^2 + c^2 n(n+1)}\, Y_n^m(\varphi, \theta).$$

Multiply both sides of the equation by the complex conjugate of $Y_{\hat{n}}^{\hat{m}}(\varphi, \theta) \sin \varphi$ and integrate with respect to θ over interval $[-\pi, \pi]$ and with respect to φ over $[0, \pi]$.

$$b_{n,m} = \frac{1}{\sqrt{\omega^2 + c^2 n(n+1)}} \int_{-\pi}^{\pi} \int_{0}^{\pi} g(\varphi, \theta) \overline{Y_n^m(\varphi, \theta)} \sin \varphi \, d\varphi \, d\theta.$$

Note that if $\omega = 0$, then $b_{0,0} = 0$.

Example

Find a formal series solution describing the radial displacement from equilibrium of a point on a flexible spherical membrane described by the initial boundary value problem below.

$$
\begin{aligned}
u_{tt} &= \Delta u \text{ for } 0 < \varphi < \pi,\ -\pi < \theta < \pi,\ t > 0 \\
u(\varphi, \theta, 0) &= \frac{1}{32} \sin^2(2\varphi) \sin(2\theta) \text{ for } 0 < \varphi < \pi,\ -\pi < \theta < \pi \\
u_t(\varphi, \theta, 0) &= 0 \text{ for } 0 < \varphi < \pi,\ -\pi < \theta < \pi
\end{aligned}
$$

Assume the unperturbed radius of the sphere is $\rho_0 = 1$.

Solution (1 of 5)

Since the initial velocity is zero, the coefficients $b_{n,m} = 0$ for all n and m. If $m \geq 0$,

$$\begin{aligned} a_{n,m} &= \frac{1}{32} \int_{-\pi}^{\pi} \int_{0}^{\pi} \sin^2(2\varphi) \sin(2\theta) \overline{Y_n^m(\varphi,\theta)} \sin\varphi \, d\varphi \, d\theta \\ &= \frac{1}{32} \int_{0}^{\pi} \sin^2(2\varphi) \sin\varphi \int_{-\pi}^{\pi} \sin(2\theta) \overline{Y_n^m(\varphi,\theta)} \, d\theta \, d\varphi \\ &= \frac{1}{32} \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} \int_{0}^{\pi} \sin^2(2\varphi) P_n^m(\cos\varphi) \sin\varphi \, d\varphi \int_{-\pi}^{\pi} e^{-im\theta} \sin(2\theta) \, d\theta. \end{aligned}$$

If $m > 0$,

$$a_{n,-m} = \frac{(-1)^m}{32} \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} \int_{0}^{\pi} \sin^2(2\varphi) P_n^m(\cos\varphi) \sin\varphi \, d\varphi \int_{-\pi}^{\pi} e^{im\theta} \sin(2\theta) \, d\theta.$$

Solution (2 of 5)

Suppose $|m| \neq 2$, then

$$\int_{-\pi}^{\pi} e^{im\theta} \sin(2\theta)\, d\theta = \left[\frac{e^{im\theta}}{m^2 - 4} \left(2\cos(2\theta) - im\sin(2\theta)\right) \right]_{\theta=-\pi}^{\theta=\pi} = 0.$$

This implies $a_{n,m} = 0$ for $m \neq \pm 2$. Thus the solution to the vibrating sphere problem takes on the form

$$\begin{aligned} u(\varphi, \theta, t) &= \sum_{n=2}^{\infty} \left[\cos(\sqrt{n(n+1)}t) \left(a_{n,-2} Y_n^{-2}(\varphi, \theta) + a_{n,2} Y_n^{2}(\varphi, \theta) \right) \right] \\ &= \sum_{n=2}^{\infty} \left[\cos(\sqrt{n(n+1)}t) \left(a_{n,-2} Y_n^{2}(\varphi, -\theta) + a_{n,2} Y_n^{2}(\varphi, \theta) \right) \right]. \end{aligned}$$

Solution (3 of 5)

Using integration by parts

$$\int_{-\pi}^{\pi} e^{\pm 2i\theta} \sin(2\theta)\, d\theta = \pm\pi\, i.$$

Let $m = 2$ then

$$a_{n,2} = -a_{n,-2} = -\frac{\pi\, i}{32}\sqrt{\frac{2n+1}{4\pi}\frac{(n-2)!}{(n+2)!}} \int_0^{\pi} \sin^2(2\varphi) P_n^2(\cos\varphi) \sin\varphi\, d\varphi.$$

Make the substitution $x = \cos\varphi$,

$$\begin{aligned} a_{n,2} = -a_{n,-2} &= -\frac{\pi\, i}{8}\sqrt{\frac{2n+1}{4\pi}\frac{(n-2)!}{(n+2)!}} \int_{-1}^{1} x^2(1-x^2) P_n^2(x)\, dx \\ &= \frac{\pi\, i}{8}\sqrt{\frac{2n+1}{4\pi}\frac{(n-2)!}{(n+2)!}} \int_{-1}^{1} x^2(1-x^2)^2 \frac{d^2}{dx^2}\left[P_n(x)\right] dx \\ &= \begin{cases} \frac{-i\sqrt{\pi}}{14\sqrt{30}} & \text{if } n = 2, \\ \frac{-i\sqrt{\pi}}{21\sqrt{10}} & \text{if } n = 4, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Solution (4 of 5)

This result further reduces the infinite series solution to the vibrating sphere problem to an exact solution containing only a finite number of terms,

$$\begin{aligned} u(\varphi,\theta,t) &= \frac{i\sqrt{\pi}}{14\sqrt{30}}\cos(\sqrt{6}t)\left(Y_2^2(\varphi,-\theta) - Y_2^2(\varphi,\theta)\right) \\ &\quad + \frac{i\sqrt{\pi}}{21\sqrt{10}}\cos(\sqrt{20}t)\left(Y_4^2(\varphi,-\theta) - Y_4^2(\varphi,\theta)\right) \\ &= \left(\frac{1}{168}\cos(\sqrt{6}t)P_2^2(\cos\varphi) + \frac{1}{420}\cos(2\sqrt{5}t)P_4^2(\cos\varphi)\right)\sin(2\theta). \end{aligned}$$

Solution (5 of 5)